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THE BELL INEQUALITIES AND THE JOINT MEASUREMENT OF INCOMPATIBLE OBSERVABLES[1]

W.M. de Muynck[2], W. De Baere[3,4], and H. Martens[2]

2. Department of Theoretical Physics
Eindhoven University of Technology
POB 513, 5600 MB Eindhoven, The Netherlands

3. Laboratory for Theoretical Physics
State University of Ghent
Proeftuinstraat 86
B–9000 Ghent, Belgium

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A quantum–mechanical theory of joint nonideal measurement of incompatible polarization observables is applied to an EPR-like experiment. It is demonstrated that this experiment yields both information satisfying and information violating the Bell inequalities. The measurement is also discussed in the context of a local hidden–variables theory. It is argued that the violation of an additional assumption of reproducibility of the hidden variable rather than violation of locality may be responsible for the violation of the Bell inequalities.

Keywords: Bell inequalities, joint measurability, hidden variables, locality.

By deriving his inequalities, John Bell [1] made it possible to distinguish experimentally between quantum mechanics and socalled local hidden–variables theories. The importance of this can hardly be overestimated, because it brought a longlasting controversy from the level of metaphysics into the domain of real experimentation. Yet, the discussion around Bell's inequalities (BI) has also had a rather odd consequence. It shifted the attention from the mutual exclusiveness of measurement arrangements of *incompatible* observables (once thought to be at the heart of the quantum mechanical problem), to the issue of nonlocality in the joint measurement of *compatible* observables on correlated and spacelike separated *individual* systems. As is well known [2] the statistics of the measurement results of quantum mechanical observables are not influenced by measurements simultaneously performed elsewhere. Therefore the nonlocality, if it would be a real effect, would have the conspiratorial property of influencing only *individual* measurement results *without having any consequence at the level of quantum mechanical expectation values.* Since this seems to bring us back to the level of metaphysics, we think the abovementioned shift to be rather unfortunate. Moreover, the conclusion of nonlocality in individual cases could only be drawn from an experimental violation of the BI if locality were the *only* assumption in its derivation. If additional assumptions could be identified in Bell's theorem, a violation of the BI might be blamed on these additional assumptions.

One such additional assumption is the existence of a joint probability distribution (jpd) $p(ijkl)$ of the four observables that are involved in the BI, which is a sufficient condition for the BI to be satisfied [3]. Here $(ijkl)$ refers to the measurement outcomes of these observables. Since among these there are incompatible pairs, no operational way of implementing such a jpd was known until recently. For this reason we had to resort to EPR–like measurements in which only two (compatible) observables are measured jointly. Due to a newly developed theory of joint measurement of incompatible observables [4,5] this restriction is now removed. In the following we shall consider an experimental arrangement for the joint measurement of the four observables, and discuss the role of the BI and its physical significance.

We first consider the joint measurement of two incompatible components of photon polarization, having spectral representations $\{E(+), E(-)\}$ and $\{F(+), F(-)\}$, respectively, first discussed by Busch [4]. The joint measurement is realized simply by means of a beam splitter (cf. Fig. 1), which either transmits the photon into the direction of the polarizer θ, or reflects it towards the polarizer θ'. If the transmission probability of the beam splitter is γ, then the detection probabilities of detectors D and D' are given by $\gamma Tr\rho E_m$ and $(1-\gamma)Tr\rho F_n$, respectively, m and n both having the two possible values "yes" and "no" corresponding with the two possible responses of the detectors[5]. The joint detection probabilities $p(mn), m, n = + \; or \; -$, for the responses of the two detectors $(+ = yes, - = no)$ are easily seen to be given by

$$p(mn) = \begin{pmatrix} 0 & \gamma Tr\rho E(+) \\ (1-\gamma)Tr\rho F(+) & 1-\gamma Tr\rho E(+)-(1-\gamma)Tr\rho F(+) \end{pmatrix}. \quad (1)$$

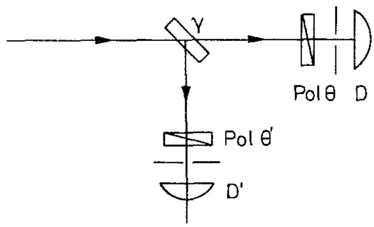

Fig. 1. Joint measurement of two incompatible polarization observables.

Calculating the two marginals of (1), we find

$$\begin{pmatrix} \sum_n p(+n) \\ \sum_n p(-n) \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 1-\gamma & 1 \end{pmatrix} \begin{pmatrix} Tr\rho E(+) \\ Tr\rho E(-) \end{pmatrix}, \qquad (2)$$

$$\begin{pmatrix} \sum_m p(m+) \\ \sum_m p(m-) \end{pmatrix} = \begin{pmatrix} 1-\gamma & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} Tr\rho F(+) \\ Tr\rho F(-) \end{pmatrix}. \qquad (3)$$

Equations (2) and (3) demonstrate in what sense the measurement can be interpreted as a joint measurement of the two incompatible components of photon polarization: the two marginals of the bivariate jpd (1) are related to the probability distributions $\{Tr\rho E(m)\}, m = +, -$, and $\{Tr\rho F(n)\}, n = +, -$, respectively, in such a way that the marginals can be considered as *disturbed* versions of the "true" probabilities $\{Tr\rho E(m)\}$ and $\{Tr\rho F(n)\}$. Indeed, by Eqs. (2) and (3), the Bohr–Heisenberg idea of complementarity is expressed, telling that in a joint measurement of incompatible observables the measurements are mutually disturbing, the disturbance being quantified by the two matrices depending on the parameter γ. We see that for $\gamma = 1$ the $\{E(m)\}$ measurement is an ideal one, no information being obtained on $\{Tr\rho F(n)\}$, whereas for $\gamma = 0$ it is the other way around. For intermediate values of γ the marginals represent nonideal information on both components, information on one component being less ideal as the other measurement is more ideal. It has been possible to derive uncertainty relations, different from the Heisenberg uncertainty relations, expressing this complementarity [5].

It is interesting to note that the relations (2) and (3) can be inverted. This implies that the probability distributions $\{Tr\rho E(m)\}$ and $\{Tr\rho F(n)\}$ can be calculated from the jpd $p(mn)$ obtained in the measurement. As a matter of fact, indicating the matrices in (2) and (3) as $(\lambda(mk))$ and $(\mu(nl))$, respectively, it turns out that the quantity

$$w(kl) = \sum_{mn} \lambda^{-1}(km)\mu^{-1}(ln)p(mn) \tag{4}$$

has all properties of a Wigner distribution on a two-dimensional Hilbert space, including the property that its marginals yield the probability distributions $\{Tr\rho E(m)\}$ and $\{Tr\rho F(n)\}$.

It will be convenient to define the positive operator-valued measure (POVM) $\{R(mn)\}$ by

$$p(mn) = Tr\rho R(mn), \tag{5}$$

and the Wigner measure $\{W(kl)\}$ by

$$W(kl) = \sum_{mn} \lambda^{-1}(km)\mu^{-1}(ln)R(mn), \tag{6}$$

the latter having the Wigner distribution (4) as its expectation value.

Next we consider an EPR-like experiment (Fig. 2) in which a correlated two-photon system is prepared as is done, e.g., in the experiments by Aspect et al. [6]. In this experiment two incompatible components of the polarization vector of each of the two photons are measured jointly in the sense discussed before (see also [7]). If we compare this experiment with Aspect's switching experiment [6], the two switching elements are now replaced by *static* semi-transparant mirrors, directing the photon to either one of the two polarizers behind each mirror (in [6] the quantities γ_1 and γ_2 are intended to be random variables that can take either the values 0 or 1). The measurement arrangement of Fig. 2 is an extension of the one of Fig. 1. It is a joint (nonideal) measurement of four observables. In this experiment these four observables coincide with the observables that are usually measured two by two in EPR-like experiments performed

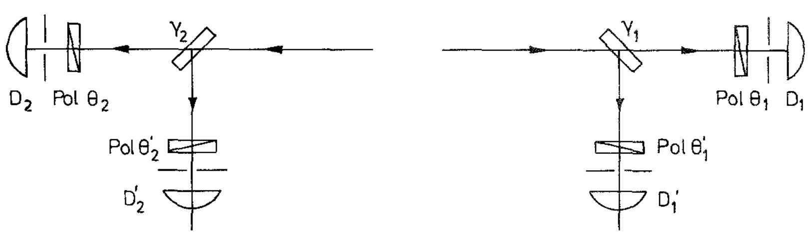

Fig. 2. EPR experiment for joint measurement of four polarization observables.

for testing the Bell inequalities like the Aspect ones. However, even if we choose both the conditions of preparation and the polarization directions such that the BI are violated in these latter tests, yet in the present experiment we find the BI to be satisfied. This must be so, because in the *joint* measurement of the *four* observables we have a quadrivariate jpd, which can easily be found to be equal to

$$p(m_1n_1m_2n_2) = Tr\rho R^1(m_1n_1)R^2(m_2n_2). \tag{7}$$

In (7) $\{R^i(m_in_i)\}, i = 1, 2$, are POVMs describing the detection rates in the two arms of the experimental arrangement. They are defined by (1) and (5), γ being replaced by $\gamma_i, i = 1, 2$.

It must be stressed that the BI are obtained here within a purely quantum mechanical context *without invoking hidden variables.* As a matter of fact, since the BI are satisfied the experiment might allow a *local* hv model. Locality is also suggested in the quantum mechanical formalism by the independence of the measurement procedures in the two arms of the interferometer, expressed by the fact that the POVM of the total measurement is a direct product of the POVMs for the two different regions:

$$R(m_1n_1m_2n_2) = R^1(m_1n_1)R^2(m_2n_2). \tag{8}$$

This seems to imply that correlations between measurement results in region 1 and region 2 must stem from the state of the object, i.e. from the preparation.

Of course, the correlations may be influenced by the mutual disturbances due to the joint measurement of incompatible observables in each arm of the interferometer. Indeed, they must be influenced because they are dependent on the parameters $\gamma_i, i = 1, 2$, the usual EPR correlations being obtained when both γ_1 and γ_2 take one of their extreme values 0 or 1.

It is remarkable that, notwithstanding the measurement disturbances in the two arms of the interferometer, the experiment also contains information on the EPR correlations. This information can be obtained from the Wigner distribution

$$w(k_1l_1k_2l_2) = Tr\rho W^1(k_1l_1)W^2(k_2l_2) \tag{9}$$

that can be obtained from the experimental jpd (7) by performing the inversion (6) in each arm of the interferometer. It is easily demonstrated that, indeed, $\sum_{l_1l_2} w(k_1l_1k_2l_2) = Tr\rho E^1(k_1)E^2(k_2)$, etc.

Comparing the present experiment and the usual EPR-like experiments like the ones by Aspect et al. [6] we observe that the differences between the measurement arrangements are of a *local* nature, it being possible to change the setup in each arm of the interferometer separately. This is expressed by the direct product structure of the POVM (8). If the essential difference between measurements either satisfying or violating the BI would be rooted in the question of (non)locality, this would mean that nonlocality is introduced by replacing the static beam splitters by time–varying ones, or by choosing γ_i equal to 0 or 1. This would add to the unobservability of the nonlocal interactions the problem of why these interactions are only effective under such special conditions.

A transition from a situation in which the BI are satisfied into one in which the BI are violated can be achieved by two separate transitions

$$R^i(m_i n_i) \to W^i(k_i l_i), i = 1, 2, \qquad (10)$$

one for each arm of the interferometer. This transition does not affect the product structure of the operator–valued measure defining the Wigner distribution (9). Therefore, locality does not seem to be threatened by the transition. An interpretation of (10) as a cancellation of the mutual disturbance of the measurement results in a joint measurement of two incompatible observables in each separate arm of the interferometer seems to be much more natural than a transition from a local to a nonlocal situation.

The (non)locality issue only arose within the context of hv theories because it was thought to be an essential presupposition for a derivation of the BI. This can be formulated in the following way in terms of the existence of a quadrivariate jpd: in the local hv theories that are usually considered the quadrivariate jpd (7) can be represented by

$$p(m_1 n_1 m_2 n_2) = \int d\lambda \rho(\lambda) p_{\gamma_1}(m_1 n_1 \mid \lambda) p_{\gamma_2}(m_2 n_2 \mid \lambda), \qquad (11)$$

in which λ is a representation of the hidden variable, $\rho(\lambda)$ is the probability of preparation and $p_{\gamma_i}(m_i n_i \mid \lambda)$ is the conditional probability of measurement results $(m_i n_i)$. In (11) the mutual disturbances must of course be taken into account in the two bivariate conditional jpds. Locality is warranted if $p_{\gamma_i}(m_i n_i \mid \lambda)$ is independent of the measurement arrangement in the other arm of the interferometer. It is important to note that different values of γ_1, γ_2 yield different jpds (11). Hence, to different EPR experiments correspond different quadrivariate jpds. If no quadrivariate jpd would exist from which the EPR probabilities can all be derived, a derivation of the BI would be impossible. However, such a jpd *does* exist if the EPR probabilities $Tr\rho E^1(m_1)E^2(m_2)$ etc. can be represented by marginals of (11) for the special choices 0 or 1 of the parameters γ_i. Then, the quadrivariate jpd

$$\int d\lambda \rho(\lambda) p_{\gamma_1=1}(m_1 \mid \lambda) p_{\gamma_1=0}(n_1 \mid \lambda)$$

$$\times\, p_{\gamma_2=1}(m_2 \mid \lambda) p_{\gamma_2=0}(n_2 \mid \lambda), \qquad (12)$$

in which in each conditional probability the correct value of γ_i is taken, would yield these EPR probabilities. So it would seem that in a hv theory locality leads to the BI after all!

Such a conclusion would be too hasty, however. It would only be a compulsory one if no additional assumptions are hidden in the assumption of the existence of the jpd (12). We believe that such additional assumptions are made! Thus, as a first remark it must be reminded that (12) does not refer to one single measurement arrangement, but is pasted together using conditional probabilities that are assumed to exist in *different experiments.* This would only be allowed if it would be possible to attribute *in an objective way* (i.e. independently of any measurement to be made) to the object in the hv state λ for every observable the probability $p(m \mid \lambda)$ of finding a value m *if the measurement is made.* In a deterministic theory in which the measurement result m is uniquely determined by the hv λ this would imply that it is possible to attribute a well-defined value of every observable (to be obtained with certainty if the measurement is actually performed) to the object in an objective, i.e. measurement-independent way.

More generally, it seems to us that the BI, being closely connected with the existence of a quadrivariate jpd, has its roots in the first place in the assumption that incompatible observables can jointly have values. In a joint measurement as discussed before this assumption may be a justified one as evidenced from the fact that the BI are satisfied here. However, things may be quite different for the usual EPR experiments. Here, in order to be able to make use of (12) it is necessary to require that the *same* value of λ is prepared in *different* EPR experiments. If, for some reason, this would be impossible, then it would be impossible to use (12) in order to derive the BI for any set of measurement results obtained in experiments that are actually performed. The necessity of such a *reproducibility hypothesis* in order to obtain the BI was stressed before by De Baere [8]. It is an assumption implicitly made in excess of the locality assumption in all derivations of the BI that we know of. Since we do not have any empirical indication of a violation of locality in EPR-like experiments, we propose that it is the reproducibility hypothesis rather than the assumption of locality that is to be blamed for the discrepancy between local hv theories and experiment.

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NOTES

1. Invited paper, International Conference on "Bell's Theorem and the Foundations of Modern Physics," Cesena, Italy, 7–10 October 1991.
4. Research associate N.F.W.O. (Belgium).
5. It is also possible to include detector efficiency in the formalism. For simplicity this is omitted here.