Compatibility of observables represented by positive operator-valued measures

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The proof of a result analogous to that in Koelman and de Muynck [Phys. Lett. A **98**, 1 (1983)] is given for the case of unbounded observables. If two, not necessarily bounded, observables are represented by a positive operator-valued measure, then the measurement of any of them is undisturbed if and only if they commute. The Naimark theorem on dilations of spectral functions is exploited. A stronger version of Wigner's theorem is given.

I. INTRODUCTION

It has been shown in Ref. 1 that the commutation of two observables can be implied by a kind of minimum principle. However, the proof given in Ref. 1 is valid only for bounded s.a. operators with discrete spectrum. In the present paper we prove an analogous result, without this restriction.

II. UNIQUENESS OF THE ORTHOGONAL SPECTRAL MEASURES

We start with a simple observation that the Kadison inequality for dilations can be extended for unbounded operators in the following sense.

Lemma 1: Let B be an unbounded s.a. operator in a Hilbert space H, let P be the orthogonal projection onto a Hilbert subspace H_1 of H, and let $A = PBP$ be a s.a. operator on H_1. Then for every $f \in D(A^2) \cap D(PB^2P)$,

$$(f|A^2 f) \leqslant (f|PB^2Pf).$$

Proof: We have $P \leqslant 1$, thus for all $g \in D(B^2)$

$$(Bg|PBg) \leqslant \|Bg\|^2 = (g|B^2 g).$$

For $h \in D(BPB) \cap D(B^2)$ we have

$$(h|BPBh) \leqslant (h|B^2 h).$$

Hence, for $f \in D(A^2) \cap D(PB^2P)$ [then $f \in D(BP)$], $Pf \in D(B^2) \cap D(BPB)$. Putting $h = Pf$, we have $h \in D(BPB)$, and thus

$$(f|(PBP)^2 f) \leqslant (f|PB^2Pf). \qquad \square$$

In a symbolic way we write

$$(PBP)^2 \leqslant PB^2P.$$

With the same notation as in Lemma 1 we have the following lemma.

Lemma 2: Let P be an orthogonal projection and B a s.a. operator. Then

$$(PBP)^2 = PB^2P \tag{1}$$

if and only if P commutes with B (i.e., P commutes with the spectral projections of B).

Proof: If P commutes with B, then obviously (1) holds.

Conversely, let (1) hold. Then $D(A^2) = D(PB^2P)$, where, as before $A = PBP$ is a s.a. operator. Let E denote the spectral measure of A and let Δ be a bounded Borel set in $\mathbb{R}^1$. Denote $A_\Delta = E(\Delta)AE(\Delta) = AE(\Delta)$. Then A_Δ is a bounded s.a. operator in H_1. Further, we have

$$A_\Delta = E(\Delta)PBPE(\Delta) = E(\Delta)BE(\Delta),$$

since $E(\Delta) \leqslant P = \mathbb{1}_{H_1}$. Thus the operator $E(\Delta)BE(\Delta)$ is bounded for every bounded Borel set Δ in $\mathbb{R}^1$. By the assumption we have

$$A_\Delta{}^2 = PE(\Delta)B^2E(\Delta)P$$
$$= E(\Delta)B^2E(\Delta) = (E(\Delta)BE(\Delta))^2,$$

that is, for every $f \in H$,

$$(f|E(\Delta)B^2E(\Delta)f) = \|BE(\Delta)f\|^2 = \|E(\Delta)BE(\Delta)f\|^2,$$

and thus $BE(\Delta)f = E(\Delta)BE(\Delta)f$.

Obviously, for every $n = 1,2,\ldots$, we have

$$B^nE(\Delta)f = B^{n-1}E(\Delta)BE(\Delta)f$$
$$= \cdots = E(\Delta)B^nE(\Delta)f = A^nE(\Delta)f.$$

Because for all bounded Borel sets Δ and all $f \in H$ the vectors $E(\Delta)f$ are analytic for the operator A, they are also analytic for the operator B. The set $\{E(\Delta)f \,|\, f \in H, \Delta \text{ bounded Borel}\}$ is dense in H_1. Thus, by the standard argument, for every $t \in \mathbb{R}^1$, we have

$$e^{itB}E(\Delta) = E(\Delta)e^{itB}P.$$

It follows that B commutes with all spectral projections of A, and, in particular, with the projection P. $\square$

As a corollary to the above lemmas we have the following proposition.

Proposition 1: Let A be an unbounded s.a. operator in a Hilbert space H, and let M be a positive operator-valued (POV) measure over the real line $\mathbb{R}^1$. Suppose that $A = \int \lambda M(d\lambda)$, where the integral converges strongly on the domain $D(A)$ of A (see Refs. 2–4). Then for every $f \in D(A^2)$

$$A^2 f = \int \lambda^2 M(d\lambda) f \tag{2}$$

if and only if M is an orthogonal spectral measure, i.e., it is a projection-valued measure over $\mathbb{R}^1$.

Proof: By the Naimark theorem[2,3] for any POV measure M there exists a Hilbert space H_0, such that $H \subset H_0$, and a projection-valued measure E_0 over $\mathbb{R}^1$, such that E_0 is the dilation of M by means of the projection P: $H_0 \to H$, i.e., for

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every $\Delta\subset\mathbb{R}^1$, $M(\Delta)=PE_0(\Delta)P$. Defining $B=\int\lambda E_0(d\lambda)$, we have $A=PBP$. Applying Lemma 2, we obtain

$$M(\Delta)=PE_0(\Delta)P=E_0(\Delta)P=E(\Delta),$$

for all Borel sets Δ in $\mathbb{R}^1$, where E is the orthogonal spectral measure of the operator A. Thus the POV measure M is identical with the spectral measure of A. □

III. THE MINIMAL SPREADING PRINCIPLE

Suppose now that A represents an unbounded observable associated with a physical system Σ. Thus we can assume that A is an unbounded s.a. operator in a Hilbert space H, affiliated with the von Neumann algebra $\mathfrak{A}$ generated by bounded observables verifiable in the system Σ. This means that all the spectral projections of A belong to the algebra $\mathfrak{A}$. Without loss of generality we can assume that $\mathfrak{A}$ is a factor. The states of the system are represented by probability measures on the lattice of orthogonal projections of $\mathfrak{A}$. By the generalized Gleason theorem[5] such measures are given by normal linear states on $\mathfrak{A}$. If $f\in D(A)$, then the integral $\int\lambda\mu_f(E_A(d\lambda))$ is well defined for the probability measure $\mu_f(Q)=(f|Qf)$, where E_A is the spectral measure of the operator A. Following this we say that a probability measure μ over the lattice of projections $\mathfrak{A}^p$ of the algebra $\mathfrak{A}$ is affiliated with the domain of A if the integral

$$\mu(A)=\int\lambda\mu(E_A(d\lambda)) \tag{3}$$

is convergent. Then we write $\mu\eta D(A)$. It is easy to see that μ can be weakly approximated by the measures $\mu_f, f\in D(A)$. It can be also shown that if $\mu\eta D(A_1)$ and $\mu\eta D(A_2)$, then $\mu(A_1+A_2)=\mu(A_1)+\mu(A_2)$.

Suppose now that $\mu\eta D(A^2)$ for a given s.a. operator A. Define

$$\sigma_0^2(A)=\mu(A^2)-\mu(A)^2. \tag{4}$$

Suppose further that there exists a POV measure M over the real line, such that

$$A=\int\lambda M(d\lambda), \tag{5}$$

where the integral converges strongly. Let us write formally

$$\sigma_M{}^2(A)=\int\lambda^2\mu(M(d\lambda))-\left(\int\lambda\mu(M(d\lambda))\right)^2.$$

Although we formulate the following results in full generality for any $\mu\eta D(A)$, it is enough to consider only measures μ_f, with $f\in D(A)$. Therefore, we omit the proofs of the following lemmas, which in the case of the measures μ_f are trivial.

Lemma 3: For every $\mu\eta D(A)$ the integral $\int\lambda\mu(M(d\lambda))$ exists and is equal to $\mu(A)$.

Lemma 4: If for two s.a. operators A_1 and A_2, we have $A_1\leqslant A_2$, then for every probability measure $\mu\eta D(A_1)\cap D(A_2)$ the following inequality holds:

$$\mu(A_1)\leqslant\mu(A_2).$$

By Lemma 1 we have always

$$A^2\leqslant\int\lambda^2M(d\lambda)=PB^2P$$

(see notation in Sec. II). Thus, for every $\mu\eta D(A^2)$, we have the inequalities of Jensen type: $\sigma_M{}^2(A)\geqslant0$, and $\sigma_0{}^2(A)\geqslant0$. Moreover, by Lemmas 3 and 4, the spreading is positive,

$$\delta(A)=\sigma_M{}^2(A)-\sigma_0{}^2(A)=\int\lambda^2\mu(M(d\lambda))-\mu(A^2)\geqslant0.$$

Thus we are ready to prove the following result.

Proposition 2: For a given observable A and for every state $\mu\eta D(A^2)$ the infimum of the values of $\sigma_M{}^2(A)$ for all possible POV measures M is achieved on the spectral measure E_A of A, i.e., $\delta(A)=0$ if and only if $M=E_A$.

Proof: If $M=E_A$, then, obviously, $\sigma_0{}^2(A)=\sigma_M{}^2(A)$ and $\delta(A)=0$.

Conversely, if $\delta(A)=0$, then, in particular, for every $f\in D(A^2)$ [we should assume $D(A^2)=D(PB^2P)$]

$$0=\mu_f(PB^2P)-\mu_f((PBP)^2)=\mu_f(PB^2P-(PBP)^2).$$

Since, by Lemma 1, $(PBP)^2\leqslant PB^2P$ in $D(A^2)\cap D(PB^2P)$, then $PB^2Pf=(PBP)^2f$, for $f\in D(A^2)$. [$D(PB^2P)$ is dense in H, cf. Ref. 2.] Hence, by Proposition 1, it follows that $M=E_A$. □

IV. INCOMPATIBLE OBSERVABLES

Finally, we apply the results obtained above to couples of observables which possess joint POV distributions.

Lemma 5: Two (not necessarily bounded) observables A and C have a joint POV probability distribution M over $\mathbb{R}^2$ [i.e., M is a POV measure over $\mathbb{R}^2$, such that $M(\mathbb{R}^2)=1$, $A=\int\lambda M(d\lambda\times\mathbb{R}^1)$, and $C=\int\gamma M(\mathbb{R}^1\times d\gamma)$], if and only if there exists a Hilbert space H_0 and two strongly commuting s.a. operators A_0 and C_0 in H_0, respectively, such that $A=PA_0P$ and $C=PC_0P$, where P is the orthogonal projection $P: H_0\rightarrow H$.

Proof: By a straightforward extension of the Naimark theorem[2] onto the case of $\mathbb{R}^2$ there exists a Hilbert space H_0 such that the POV measure M has a dilation to an orthogonal spectral measure E_0 over $\mathbb{R}^2$, with respect to the projection $P: H_0\rightarrow H$. Since the marginal measures $E_0(\mathbb{R}^1\times\cdot)$ and $E_0(\cdot\times\mathbb{R}^1)$ are projection valued, then the operators $A_0 := \int\lambda E_0(d\lambda\times\mathbb{R}^1)$ and $C_0 := \int\gamma E_0(\mathbb{R}^1\times d\gamma)$ are s.a. operators and mutually commuting. The result easily follows. □

Proposition 3: Let A and C be two observables in the system Σ which have a joint POV probability distribution M. Suppose that for every state $\mu\eta D(A^2)$ of the system Σ we have $\sigma_0{}^2(A)=\sigma_M{}^2(A)$. Then A and C commute.

Proof: By Lemma 5 there exists a Hilbert space H_0, such that $H\subset H_0$, and A and C can be dilated to two commuting operators A_0 and C_0 in H_0. By Proposition 2, it follows from the assumption that

$$E_A=M(\cdot\times\mathbb{R}^1).$$

It is easy to see that since for all Borel sets Δ, $\Delta'\subset\mathbb{R}^1$,

$$E_0(\Delta\times\mathbb{R}^1)E_0(\mathbb{R}^1\times\Delta')=E_0(\mathbb{R}^1\times\Delta')E_0(\Delta\times\mathbb{R}^1),$$

and

$$E_A(\Delta)=E_0(\Delta\times\mathbb{R}^1)P,$$

we have

$$E_A(\Delta)f\in D(C) \text{ and } CE_A(\Delta)f = E_A(\Delta)Cf$$

for every $f\in D(C)$.

In particular, by the argument similar to that in Lemma 2 concerning analytic vectors, we get the commutation of $E_A(\Delta)$ with the spectral projections of the operator C. Thus A and C commute. ☐

Proposition 3 is a stronger version of Proposition 6.1, Chap. II, in Ref. 3. There the commutativity is derived from the minimal spreading of both observables. In the above Proposition 3 one of the marginal distributions can be a genuine POV measure, not necessarily a projection-valued one.

The above result can be interpreted as follows. In orthodox quantum mechanics only observables corresponding to mutually commuting self-adjoint operators are believed to be jointly measurable. The Naimark theorem, as exploited in Lemma 5, permits us to extend the notion of joint measurability to so-called incompatible observables which correspond to noncommuting s.a. operators. Thus, the joint measurement of A_0 and C_0 described in Lemma 5 can be interpreted as a joint measurement of A and C, with

$$M(d\lambda\times\mathbb{R}^1) = PE_0(d\lambda\times\mathbb{R}^1)P,$$

$$M(\mathbb{R}^1\times d\gamma) = PE_0(\mathbb{R}^1\times d\gamma)P.$$

It follows that two incompatible observables are jointly measurable whenever they have a joint POV probability distribution in the sense of Lemma 5. The question whether any pair of incompatible observables allows a joint POV probability distribution, and, hence, is jointly measurable, is not discussed here. For the position $\hat{q}$ and momentum $\hat{p}$ many examples are well known.[3,6,7]

The notion of a POV measure makes it possible to give a generalized definition of a quantum mechanical measurement of incompatible observables. Our present results extend the results of Ref. 1 onto the general case of unbounded observables, including position and momentum. It has been demonstrated that it is possible to measure jointly these observables only when a mutual disturbance is allowed, affecting the second (and higher) moments of both distributions. As a corollary to this result we prove a stronger version of Wigner's theorem.[8,9]

Corollary 1: There is no phase-space representation of quantum mechanics which satisfies all of the following three requirements: (i) the distribution function $f(q,p)$ on $\mathbb{R}^2$ of a state with the density operator ρ is the expectation value of a s.a. operator $M(q,p)$ (defining a POV measure over $\mathbb{R}^2$), i.e.,

$$f(q,p) = \mathrm{Tr}[\rho M(q,p)];$$

(ii) $f(q,p)\geqslant 0$;

and (iii) either

$$\int_{-\infty}^{\infty} f(q,p)dp = (q,\rho q) \tag{6}$$

or

$$\int_{-\infty}^{\infty} f(q,p)dq = (p,\rho p).$$

Proof: The impossibility of the phase-space representation is implied by Proposition 3, because the operators $\hat{q} = \int qM(q,p)dq\,dp$, and $\hat{p} = \int pM(q,p)dq\,dp$ would commute if the conditions (i)–(iii) were satisfied. ☐

Notice that in the previous version of Wigner's theorem[8,9] both conditions in (iii) are required. A similar result was obtained by Twareque Ali and Prugovečki[10] under the additional hypothesis of covariance under the Galilean group.

Corollary 2: (iii)′ The operator which corresponds to a phase-space function of the form $A(q) + B(p)$, where $A(q)$ and $B(p)$ are arbitrary Borel functions, is $A(\hat{q}) + B(\hat{p})$, thus

$$A(\hat{q}) + B(\hat{p}) = \int\{(A(q) + B(p))M(q,p)\}dq\,dp \tag{7}$$

implies both conditions in (iii), and hence, it is not a weaker condition, as assumed in Ref. 9.

Proof: According to Proposition 1, in order to prove (6), it is sufficient to take $A(q) = 1, q$, and q^2, and $B(p) = 0$ in (7). ☐

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