The Uncertainty Principle in a QND Measurement of Photon Number

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Abstract

We study a simple setup for non–destructive detection of photon number. We in particular compare the device's inaccuracy and its phase disturbance to the uncertainty principle. As the traditional uncertainty principle involves neither inaccuracy nor disturbance, a new type of uncertainty relation is employed. Self–phase modulation is taken into account, losses are neglected.

Introduction

Quantum optics makes a number of fundamental quantum measurements feasible. Heisenberg's γ–microscope(1) is such an experiment. Heisenberg intended to show how a position measurement in quantum mechanics must disturb momentum. The size of the disturbance is inversely proportional to the inaccuracy of the position meter. A non–destructive measurement of photon number via the optical Kerr–effect may be seen as an analog of Heisenberg's γ–microscope. Such a Kerr–device would then be expected to disturb phase, the variable conjugate to photon number.

The setup in its simplest form is sketched in fig. 1. A probe beam P, initially in a coherent state $|\alpha\rangle_P$, and a signal beam S (frequency ω_S) are passed through a medium

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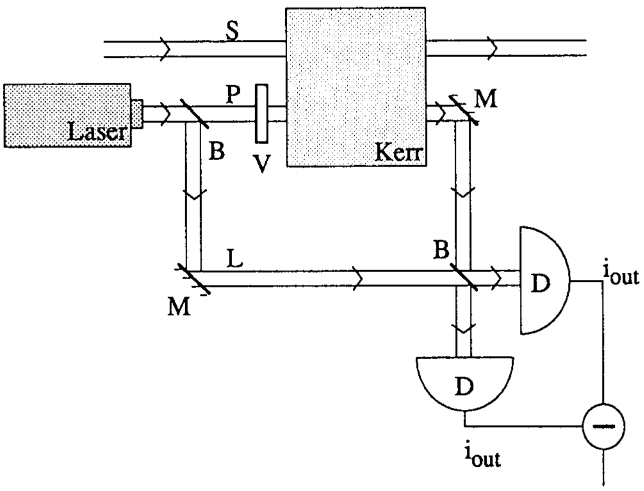

fig. 1 *Kerr QND measurement of photon number. A signal beam* S *is mixed with a probe beam* P *into a non-linear medium. The outgoing probe beam is fed into a homodyne detector. [*V *is a delay,* B*'s are beamsplitters,* M*'s are mirrors,* D*'s are detectors, and* L *indicates the homodyne local oscillator beam.]*

with a third order non–linearity. During a time $\tau = \ell/c$ the two modes interact via a Hamiltonian

$$H_I = 4\chi_{SP} N_S N_P + \chi_{PP} {N_P}^2 + \chi_{SS} {N_S}^2 \quad ; \tag{1}$$

with ($\hbar = 1$)

$$\chi_{SP} := \frac{9}{8V\epsilon^2}\, \omega_S \omega_P\, \chi^{(3)}(\omega_P;\, \omega_P, -\omega_S, \omega_S) \;\; ; \quad N_S := a_S^\dagger\, a_S \quad .$$

The operator a_S is the annihilation operator for the S mode. N_P, a_P, χ_{SS} and χ_{PP} are defined analogously. After the interaction the probe beam P is detected in a homodyne detector. This may be seen[2] as a measurement of

$$Q \;:=\; \tfrac{1}{2}\sqrt{2}\; a_P \exp(-i\varphi) + \text{h.a.} \quad . \tag{2}$$

The phase φ can be controlled *via* the delay V.

Photon Number

The setup of fig. 1 is intended to measure photon number "inaccurately" [3]. But in what sense? We first look at the device without self–phase modulation, i.e., we neglect the χ_{PP} and χ_{SS} terms in (1). Then, tracing over the P Hilbert space,

$$\langle Q(\tau)\rangle \;=\; \sqrt{2}\;|\alpha|\;\cos(4\tau\chi_{SP}N_S + \varphi) \quad . \tag{3}$$

More generally, the q outcome distribution can be written as

$$P(\mathrm{d}q) \;=\; \sum_n f_n(\mathrm{d}q)\, P_n \quad ; \tag{4}$$

with $f_n(\Delta q) \geq 0$ and $\int f_n(\mathrm{d}q) = 1$. Here

$$f_n(\mathrm{d}q) \;=\; \frac{\mathrm{d}q}{\sqrt{2\pi}\;\sigma}\exp\left[-\frac{[q - \sqrt{2}|\alpha|\cos(4\tau\chi_{SP}n + \varphi)]^2}{2\sigma^2}\right] \quad ; \tag{5}$$

$$P_n \;=\; {}_S\langle n|\rho_S|n\rangle_S \quad ; \quad \sigma^2 \;=\; \tfrac{1}{2} \quad .$$

The function $f_n(\mathrm{d}q)$ may be thought of as a conditional probability distribution, giving the probability of outcome q given the accurate value n. The q outcome distribution is the N_S distribution smeared by means of $f_n(\mathrm{d}q)$ [4]. This is a consequence of the form of the interaction Hamiltonian (1): H_I is a function of the operator N_S, which is conserved[5]. Thus (4) is a relation between *probability distributions*, and not only between *expectation values*. The measurement is 'inaccurate' in the sense of (4).

If we assume that S contains relatively few photons, i.e.

$$4\tau\chi_{SP}n \ll 1 \quad , \tag{6}$$

we can linearize (5), so that it becomes a convolution Gaussian. Then f is a function of q–n alone: the measurement is *covariant.* Moreover, (4) can be inverted (deconvolution): the *whole* N_S distribution can be estimated using the Kerr–device outcomes.

Looking at (4), we see that the amount of inaccuracy of the device is characterized by the n–width of the likelihood-type function $f_n(\mathrm{d}q)$ for fixed q. Equivalently, we can characterize the amount of inaccuracy by the distinguishability of the distributions $f_n(\mathrm{d}q)$ and $f_{n+1}(\mathrm{d}q)$. Given the form (5) of the smearing function f, the width or distinguishability can be characterized by the noise–gain ratio:

$$\delta_N := \langle \Delta^2 Q(\tau)\rangle^{\frac{1}{2}}/G \quad ; \tag{7}$$

with

$$\langle \Delta^2 Q(\tau)\rangle := \langle (Q(\tau) - \langle Q(\tau)\rangle)^2\rangle = \tfrac{1}{2} \quad ; \tag{8}$$

$$G := \left| \frac{\partial \langle Q(\tau)\rangle}{\partial N_S} \right| = 4\sqrt{2}\, \tau\chi_{SP}\, |\alpha \sin(\varphi + 4\tau\chi_{SP} N_S)| \quad . \tag{9}$$

In the low photon number regime the inaccuracy is therefore given by

$$\delta_N \simeq \frac{1}{8\tau\chi_{SP}\, |\alpha\, \sin(\varphi)|} \quad . \tag{10}$$

The delay phase φ is to be chosen such that accuracy is optimal[6,7]. In this simplest case the optimal choice is $\varphi_{opt} = \pi/2$.

If we include self-phase modulation, (4) is still valid. The smearing function f does not have the form (5) anymore, however. Nevertheless the approximate covariance of (5) can be expected to remain intact for low non–linearities[6]. Thus the inaccuracy characterization via noise/gain ratio remains adequate. We get ($\tau\chi_{PP} \ll 1$)

$$\langle Q(\tau)\rangle \simeq \sqrt{2}\, \gamma |\alpha| \cos(4\tau\chi_{SP} N_S + \varphi') \quad ; \tag{3'}$$

$$\langle \Delta^2 Q(\tau)\rangle \simeq \tfrac{1}{2} + |\alpha|^2 (1-\gamma^2) + \\ + |\alpha|^2 \gamma^2 [\gamma^2 \cos(8\tau\chi_{SP} N_S + 2\varphi' + \varphi_0) - \cos(8\tau\chi_{SP} N_S + 2\varphi')] \quad ; \tag{8'}$$

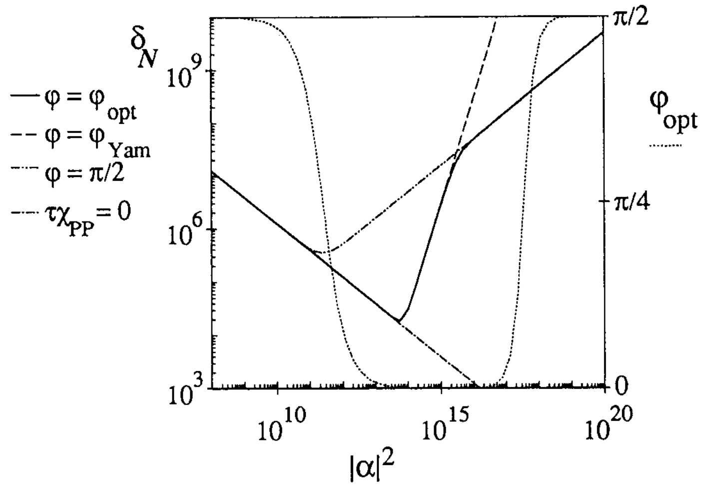

fig. 2 *N–inaccuracy [left y-axis] vs. laser power for different choices of phase φ [φ is controllable by* V *in fig. 1]. Note that, even for the optimal choice φ_{opt} [dotted, right y–axis] for the phase, there is a lower bound to the accuracy achievable.* [$\cot\mathrm{an}(\varphi_{\mathrm{Yam}}) = 4\tau\chi_{\mathrm{PP}}|\alpha|^2$ *(ref. 6)*; $\tau\chi_{\mathrm{PP}} = \tau\chi_{\mathrm{SP}} = 10^{-12}$]

with

$$\gamma = \exp(-2(|\alpha|\tau\chi_{\mathrm{PP}})^2) \quad ;$$

$$\varphi' = \varphi - \gamma\tau\chi_{\mathrm{PP}} - \gamma|\alpha|^2 \sin(2\tau\chi_{\mathrm{PP}}) \quad ;$$

$$\varphi_0 = 2\tau\chi_{\mathrm{PP}} - 8|\alpha|^2 (\tau\chi_{\mathrm{PP}})^3 \quad .$$

Assuming the low photon number regime, noise is minimal if $\varphi' \simeq \pi/4$ [ref.6, fig. 26]. Gain is maximal, however, if $\varphi' = \pi/2$. Thus optimal accuracy requires a trade-off between noise and gain. But even for optimal phase φ_{opt} (determined numerically; fig.2, dotted line), performance is limited. This deterioration of measurement quality is a consequence of the effect of self–phase modulation on the P–state. Initially coherent, it becomes extended in the phase direction ("crescent squeezing" [6]). Minimal inaccuracy $\delta_{N,\mathrm{opt}} \sim (\tau\chi_{\mathrm{PP}})^{3/5}(\tau\chi_{\mathrm{SP}})^{-1}$ is achieved for probe strength $|\alpha|_{\mathrm{opt}} \sim (\tau\chi_{\mathrm{PP}})^{-3/5}$ (fig.2, fig.3). Low inaccuracies require high non-linearities.

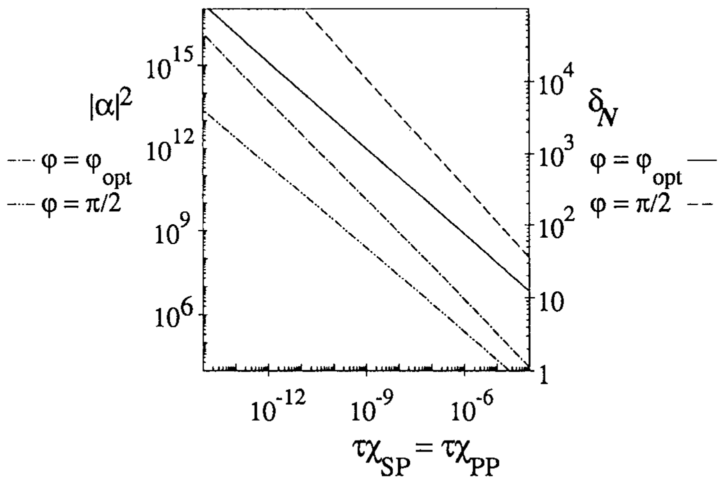

fig. 3 *N–inaccuracy [right y-axis] at optimal laser power* $|\alpha|^2$ *[left y–axis], for* $\varphi = \pi/2$ *and* $\varphi = \varphi_{\rm opt}$ *, vs. non-linearity* $\tau\chi_{\rm PP} = \tau\chi_{\rm SP}$*. Large coefficients of non-linearity are necessary for low inaccuracies.*

Phase

As stated in the introduction to the paper, we want to investigate the device's performance compared to the uncertainty principle, analogously to Heisenberg's γ–microscope. Thus we study phase in the Kerr–device. To describe phase we choose[8] the "states"

$$|\phi\rangle \;:=\; (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{N}} \exp(\mathrm{i}\phi n)|n\rangle \quad . \tag{11}$$

They are not orthogonal, but do satisfy

$$\int_{-\pi}^{\pi} |\phi\rangle\langle\phi|\,\mathrm{d}\phi \;=\; 1 \quad . \tag{12}$$

Eq. (12) implies that $\{|\phi\rangle\langle\phi|\mathrm{d}\phi\}$ is a *positive operator–valued measure*[9] (POVM). The phase states are eigenstates of the operator

$$e := \sum_{n\in\mathbb{N}} |n\rangle\langle n+1| \quad . \tag{13}$$

We prefer this phase representation (due to Lévy–Leblond) over the more well-known Carruthers & Nieto phase description[10], because of the clumsiness of the latter. Here we have a nice Fourier relation (viz. (11)) between phase and photon number, as well as a Weyl–type commutation relation

$$e^a \exp(ibN) = \exp(ibN)\, e^a \exp(iab) \quad (a\in\mathbb{Z},\ b\in\mathbb{R}) \quad . \tag{14}$$

We look again first at the case without self–phase modulation. If we would want to know the signal's phase as it was before the interaction with the probe beam, we must try to compensate for this interaction as much as possible. This compensation cannot be complete, however. We can measure on the S beam emerging from the Kerr device $\{|\phi+\phi_0\rangle\langle\phi+\phi_0|\,d\phi\}$. This POVM becomes (Heisenberg picture)

$$|\phi+\phi_0\rangle_S\langle\phi+\phi_0|\,d\phi \longrightarrow \int_{-\pi}^{\pi} d\phi'\ |\phi'\rangle_S\langle\phi'|\ \mu(d\phi,\phi') \quad ; \tag{15}$$

with phase bias given by

$$\phi_0 = 4|\alpha|^2\tau\chi_{SP} - \tau\omega_S \quad .$$

Here, again, $\int \mu(d\phi,\phi') = 1$ and $\mu(\Delta\phi,\phi') \geq 0$ (cf.(4)). Thus measurement of $\{|\phi+\phi_0\rangle\langle\phi+\phi_0|\,d\phi\}$ on the outgoing signal mode would be interpretable as a smeared measurement of initial phase. An accurate measurement of initial phase has become impossible. Phase is *disturbed*[5]. The amount of disturbance is given by the lowest inaccuracy with which initial phase can conceivably be determined. Here that is the width of μ in (15). In the low photon number regime,

$$\mu(d\phi,\phi') \simeq \theta_3\,[\tfrac{\phi-\phi'}{2};\ \exp(-8|\alpha\tau\chi_{SP}|^2)]\ d\phi \quad . \tag{16}$$

θ_3 denotes the third of Jacobi's theta functions. Given (16)'s covariance, a suitable measure for phase disturbance is[9]

$$V_\phi := -1 + \left|\int_{-\pi}^{\pi} \mu(d\phi,0)\exp(i\phi)\right|^{-2} \quad . \tag{17}$$

Substituting (16) in (17), we get

$$V_\phi \;=\; -1 + \exp(2|\alpha|^2\,[1 - \cos(4\tau\chi_{SP})]) \quad . \tag{18}$$

If we include self–phase modulation, (15) turns into

$$|\phi+\phi_0'; -2\tau\chi_{SS}\rangle_S\langle\phi+\phi_0'; -2\tau\chi_{SS}|\,d\phi \;\longrightarrow\; \int_{-\pi}^{\pi} d\phi'\; |\phi'\rangle_S\langle\phi'|\; \mu(d\phi,\phi') \quad ; \tag{15'}$$

with μ as in (15), and

$$\phi_0' \;=\; \phi_0 + \tau\chi_{SS} \quad ;$$

$$|\phi;\nu\rangle \;:=\; (2\pi)^{-\frac{1}{2}} \sum_{n\in\mathbb{N}} \exp(i\phi n + \tfrac{1}{2}i\nu n(n+1))\;|n\rangle \quad . \tag{11'}$$

Self-phase modulation *distorts* phase into a slightly different degree of freedom. But, like the phase bias ϕ_0, this distortion can be compensated for in principle. By measuring after the Kerr–device the POVM corresponding to (11'), we obtain information precisely about phase (11). Thus phase *disturbance,* which cannot be compensated for, is unaffected by self–phase modulation.

Uncertainty principle

Intuitively, it is clear that phase disturbance and N measurement inaccuracy are complementary[1]. But the traditional uncertainty principle (either in its Heisenberg or in its Robertson form), has nothing to say about the subject. It bounds statistical *scatter* in *independent, accurate* measurements[11]. Inaccuracy is not involved. The measurement transformation, essential for the 'disturbance' concept, is not at all taken into account. As indicated above, the phase is disturbed in the sense that it has become impossible to measure accurately the pre–measurement phase. The device consisting of the non–destructive N meter followed by a measurement of initial phase, would be a joint phase–number meter. Therefore an uncertainty relation limiting the accuracy achievable in joint measurements of incompatible observables, also limits the phase-disturbance in a non–destructive photon number measurement[5]. Such uncertainty relations have become available only relatively recently, however[12,13,14]. For the

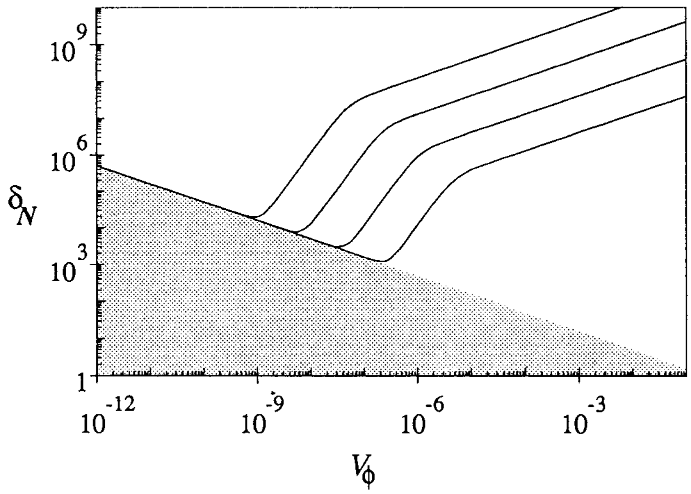

fig. 4 *Possible combinations of N–inaccuracy and phase disturbance for the Kerr-device with self-phase modulation (the lines correspond to* $\tau\chi_{PP} = \tau\chi_{SP} = 10^{-12}, 10^{-11}, 10^{-10}, 10^{-9}$*). The combinations forbidden by relation (19) are in the shaded area.*

phase-number case an *inaccuracy inequality* can be derived for covariant measurements[15] (cf. ref. 8). For the inaccuracy-disturbance complementarity we study here, it yields the theoretical limit

$$\delta_N^2 \, V_\phi \;\geq\; \tfrac{1}{4} \quad . \tag{19}$$

Since the measurement here can be considered approximately covariant [see (16) and (5)], the bound (19) is suitable to compare the Kerr–device to. In the simple case that we have no self–phase modulation, the inaccuracy and disturbance can be seen to satisfy ($\tau\chi_{SP} \ll 1$)

$$\delta_N^2 \, \log(1 + V_\phi) \;=\; \tfrac{1}{4} \quad , \tag{20}$$

which is slightly above (19). If we include self–phase modulation the performance deteriorates, as was to be expected from (8′). This is indicated in fig. 4.

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