Towards a new uncertainty principle: quantum measurement noise

Hans Martens and Willem M. de Muynck

Department of Theoretical Physics, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

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Two generalizations of a known approach to the joint measurement of position and momentum to the joint measurement of more general pairs of observables are compared. They weaken the restrictions on "noisy" measurements that prevented the above method from being more generally usable, in two different ways: additive object-dependent noise versus object-independent non-additive noise. In the latter approach a lower bound for the amount of noise in a joint measurement of incompatible observables is found, not as a consequence of the usual Heisenberg scatter principle, but of a new "inaccuracy principle". Physically realizable examples are given.

1. Introduction

Quantum measurement theory has in recent years become practically relevant, particularly in the area of quantum optics. Of special interest are quantum limits to measurement accuracy. These arise as a consequence of the impossibility to jointly measure incompatible observables accurately. A standard way of dealing with the joint measurement problem has been known for some time [1–8] for the case of position–momentum. In the following we shall first give a concise review of these results. They can be generalized in different ways, depending on the notion of "measurement" that is employed. In the following two such generalization schemes are discussed. The first relies on a characterization of measurement through expectation values (requirement of unbiasedness of the measurement). It will be shown that in this approach noise due to object fluctuations, i.e. to the preparation procedure, cannot be distinguished from noise in the measurement device. Strictly speaking, only the latter type of noise is related to measurement inaccuracy. A second measurement characterization scheme involves a strict separation of the two types of noise from the beginning. In this way can the latter scheme, applied to the joint measurement problem, lead to a limit for only the measurement noise. As only this noise is related to measurement inaccuracy, this type of limit is a true quantum limit to measurement accuracy. The usual Heisenberg uncertainty relations do not involve measurement accuracy, and thus imply no such quantum limit.

Finally, we shall compare the applicability of the two methods. Despite the fact that the latter scheme is the more restrictive one, its applicability appears to be not more limited. This is evidenced by an application to joint measurements of quantities other than position–momentum, the latter being essentially the only case treated in the literature thus far.

2. Standard approach: position–momentum

Consider two quantum systems, labeled o and a. They represent the object and an ancillary system, respectively. The operators on these systems are identified by corresponding indices. The joint state of object and ancilla just prior to the beginning of the measurement (on \( \mathbb{H}_o \otimes \mathbb{H}_a \)) is given by

\[
\rho = \rho_o \otimes \rho_a .
\]  

Here \( \rho_a \) is the initial state of the ancilla. We want to measure an object observable represented by \( R_o \otimes 1_a \). To achieve this, we let ancilla and object interact for a time \( \tau \). After the interaction is completed, a "read-
out observable" $R'$ is measured (Heisenberg picture).

$$R'(t) := R_o \otimes 1_a + G^{(r)}.$$  

(2)

Alternatively, we can let an $R'$ measuring device interact with both ancilla and object $^1$. The operator $G^{(r)}$ represents the deviations of the measurement results from the "true" values, it represents noise. In the standard approach, the noise operator $G^{(r)}$ is taken to be an ancilla operator, viz.

$$G^{(r)} = 1_o \otimes G_a^{(r)}.$$  

(3)

The setup was intended to measure $R_o$ "inaccurately". Thus, to warrant this interpretation, the $R'$ measurement is required to be unbiased with respect to $R_o$:

$$\forall \rho_o \quad \text{Tr} [\rho_o \otimes \rho_a R'(\tau)] = \text{Tr}(\rho_o R_o) ,$$

$$\forall \rho_o \quad \text{Tr}(\rho_o \otimes \rho_a G^{(r)}) = 0.$$  

(4)

The $R'$ probability distribution can be written according to

$$P_{R'(\tau)} (dr') = \text{Tr}_o [\rho_o M_{R'(\tau)} (dr')] ,$$

$$M_{R'(\tau)} (dr') = \text{Tr}_a [\rho_a E_{R'(\tau)} (dr')] .$$

Here $E_{R'(\tau)} (dr')$ is the spectral or projection valued measure (PVM) of $R'(\tau)$ on $\mathcal{H}_o \otimes \mathcal{H}_a$. Thus we can reduce a PVM on $\mathcal{H}_o \otimes \mathcal{H}_a$ to a positive operator valued measure (POVM) [1–3] on $\mathcal{H}_a$, viz. $M_{R'(\tau)} (dr')$. The POVM $M$ forms a representation of the noisy measurement, alternative to $R'$ itself. POVMs represent a more general notion of observable than Hermitian operators. They satisfy

$$\forall \Delta r' \quad 0 \leqslant M_{R'(\tau)} (\Delta r') , \quad 1 = M_{R'(\tau)} (\mathbb{R}) .$$

To cope with noisy measurements, it is necessary to work in this generalized frame, as also stressed by Yuen [9].

From (4) it follows directly [10,11] that the $R'(\tau)$ probability distribution is broader than the $R_o$ one. The variance of the former distribution can be considered as the sum of two positive contributions, one stemming from the (ideal) $R_o$ distribution, the other one being caused by the excess noise: using (2) we get

$$\langle \Delta^2 R'(\tau) \rangle = \langle \Delta^2 R_o \rangle + \langle \Delta^2 G^{(r)} \rangle .$$  

(5)

Relation (5) can of course also be phrased in terms of $M_{R'(\tau)}$, rather than $R'$.

Moreover, (2) and (3) imply that the $R'(\tau)$ outcome distribution can be derived from the $R_o$ outcome distribution by convolution with the $G_a^{(r)}$ distribution:

$$\forall \rho_o \quad P_{R'(\tau)} (dr') = \int_{-\infty}^{\infty} P_{R_o} (dr) g(dr', r) ,$$

with

$$g(dr', r) = \tilde{g}(r' - r) dr' = p_{G_a^{(r)}} (r' - r) dr'$$  

(6')

(6)

$P$'s denote probabilities, $p$'s probability densities.

We may equivalently write

$$M_{R'(\tau)} (dr') = \int_{-\infty}^{\infty} E_{R_o} (dr) g(dr', r) ,$$

(7)

$E_{R_o} (dr)$ being the PVM of $R_o$ on $\mathcal{H}_a$. The function $g$ in (7) characterizes the "smearing" in the $R'(\tau)$ distribution with respect to $R_o$. The function $g$ is the $G_a^{(r)}$ probability distribution, and therefore a property of the measuring device (i.e. the ancilla) alone. The function $g$ depends only on $\rho_a$, not on the initial object state $\rho_o$. Thus condition (3) defines object-independent noise.

The approach (2)–(4) can be used as a starting point for a joint measurement of two observables represented by the non-commuting Hermitian operators $R_n$ and $T_o$ on $\mathcal{H}_o$. The aim is to find two noise operators $G_a^{(r)}$ and $G_a^{(t)}$ on $\mathcal{H}_a$ such that $R'(\tau) = R_o + G^{(r)}$ and $T'(\tau) = T_o + G^{(t)}$ commute $^2$. Next, a non-trivial lower bound to the amounts of excess noise, as defined in (5), is to be derived. But this approach faces one major handicap (cf. ref. [9]). Since

$^1$ The so-called "image band" plays the role of ancilla in this sense in the heterodyne detector [8].

$^2$ We might also look for two functions $\tilde{g}^{(r)} (r' - r)$ and $\tilde{g}^{(t)} (t' - t)$ such that the POVMs $M_{R'(\tau)} (dr')$ and $M_{T'(\tau)} (dr')$ are compatible, i.e. are marginals of a common POVM.
\[ [R'(\tau), T'(\tau)] = 0, \]

\[ [R_0, T_0] = 1 + [G^{(r)}, G^{(i)}] = 0, \quad (8) \]

compatibility (8) can only be valid if \([R_0, T_0] = ic1,\) i.e. only for position-momentum like pairs. For such pairs \([1-8],\) as follows immediately from (8), (5) and the Heisenberg inequality for the noise operators, the excess variances satisfy

\[ \langle A^2 G^{(r)} \rangle \langle A^2 G^{(i)} \rangle \geq \frac{1}{4} c^2. \quad (9) \]

Despite its limited usefulness, this formal joint measurement procedure is relevant in certain experiments: both (balanced) heterodyning \([12]\) and parametric amplification of “position” and “momentum” \([13]\) give rise to excess noises describable by (2)-(4).

3. Expectation-value based approaches

If we want to generalize (9), we must first find a less restrictive characterization of “noisy” measurements. The set of conditions additivity (2), object-independence (3) and unbiasedness (4), which formed that characterization in the standard approach, has to be weakened. We saw in the previous section that two ways of viewing the noise in the standard approach exist. Considering object-independence (3) as primary, the smearing (7) described the noise. On the other hand, from the point of view of unbiasedness (4), the excess noise defined in (5) formed such a description. Thus, depending on one’s attitude towards “measurement”, different possibilities of generalizing the standard approach arise. If a measurement is characterized by its expectation values, indeed often an important result, (4) should be maintained. Accordingly, (3) should be dropped as a general requirement. The unbiasedness criterion (4) has been widely used, e.g., by de Muynck et al. \([10,14],\) Busch \([15]\) and Schroек \([16]\). Arthurs and Goodman \([17]\), working along these lines, generalized (9) to

\[ \langle A^2 G^{(r)} \rangle \langle A^2 G^{(i)} \rangle \geq \frac{1}{4} \langle [R_0, T_0] \rangle^2. \quad (10) \]

Yuen \([9]\) also intends to generalize inequality (9) using an expectation-value based approach. He starts from a particular joint measurement POVM for position and momentum, viz.

\[ M_{\alpha} \langle d\alpha, d\alpha' \rangle = |\alpha \rangle \langle \alpha | \frac{d^2 \alpha}{\pi}, \]

\[ \alpha = \frac{1}{2} \sqrt{2} (q' + ip'). \quad (11) \]

Here \(|\alpha \rangle\) are the harmonic oscillator coherent states. This POVM results from the joint PVM of \(P' = P_0 + P_a\) and \(Q' = Q_0 - Q_a\) on \(\mathcal{H}_a \otimes \mathcal{H}_a\) by taking \(\rho_a = |0 \rangle \langle 0|\) (the harmonic oscillator ground state). These coherent states form the (overcomplete) set of eigenstates of the (non-Hermitian) annihilation operator \(\alpha = Q + iP\). Yuen then proposes to generalize this by looking for a POVM generated by the eigenstates of the operator \(B_o = R_0 + iT_0\). But note that the joint PVM of \(Q'\) and \(P'\) on \(\mathcal{H}_a \otimes \mathcal{H}_a\) has two arguments, reflecting the fact that in a joint measurement every measurement outcome consists of two numbers. Consequently, if the POVM on \(\mathcal{H}_a\) is to be derived from a set of states, this set must be bivariate. It is unclear to what extent the operator \(B_o\) can have a bivariate set of eigenstates if \(R_0\) and \(T_0\) are not the position and momentum operators. This makes it doubtful whether the generalization Yuen achieves in this way is indeed substantial.

But even in the Arthurs and Goodman version, the expectation-value based approach has a number of serious drawbacks. Noise processes in measurements in general violate eq. (4). If they do, it may appear that this can be remedied by simply subtracting the bias from the measurement result. But by dropping (3), the noise operators are allowed to act on the full product Hilbert space. Therefore the bias is generally not a constant: it may differ for different initial object states \(\rho_o\). Then bias subtraction is impossible. Thus, from a physical point of view, unbiasedness constitutes a substantial idealization. Moreover, also the excess variance \(\langle A^2 G^{(r)} \rangle\) and, more generally, the connection between \(R_0\) and \(R'\) probability distributions, will in general depend on \(\rho_o\). Consider as an example a two-dimensional particle, with position operators \(Q_{1,0}\) and \(Q_{2,0}\). In order to measure \(Q_{1,0}\), we couple it with an ancilla such that

\[ Q'_1(\tau) = Q_{1,0} \otimes 1_s + Q_{2,0} \otimes Y_s. \quad (12) \]

for some ancilla operator \(Y_s\). The ancilla state is such that \(\text{Tr}(\rho_o Y_s) = 0\). Then the measurement is indeed
unbiased with respect to $Q_{1,0}$. But the difference between measurement outcome and true value is here described by the noise operator $Q_{2,0} \otimes Y_\alpha$. It is therefore clear that the noise will also depend on $\rho_0$. As the example shows, $\langle \Delta^2 G^{(r)} \rangle$ is an inaccuracy characterization that, contrary to common practice, cannot always be seen as a property of the measurement device alone. In the excess variance bound (10), the fact that the noise now in general depends on $\rho_0$, results in a bound that depends on $\rho_0$, too (unless $[R_0, T_0] - \sim 1$; cf. (9)). In one of the examples treated below we shall meet this dependence again.

4. Non-ideality approach

We saw in the previous section that, within an expectation-value based approach, the notion of inaccuracy as purely a property of the measuring instrument has to be abandoned on behalf of general applicability. Its reliance on the labeling of the measurement scale forms a second drawback. The outcome set of the POVM (~ spectrum of the self-adjoint operator on the product Hilbert space), which contains this labeling, is merely a matter of convenience [18], however. Nothing physical in the measurement device is changed if we alter its scale. Therefore an acceptable “inaccuracy” notion should be insensitive to the meter labeling. A third disadvantage of an expectation-value based approach is that the expectation value is not always the only parameter of a probability distribution we are interested in. In dropping (3) we, however, in general cut ourselves off from obtaining information about the $R_0$ distribution other than $\langle R_0 \rangle$.

Accordingly it seems worthwhile to consider another approach. The first objection to the expectation-value based approach can be countered by maintaining the object-independence condition (3) and dropping (4). The connection between the noisy measurement and the ideal can then be saved by replacing additivity (2) by

$$R'(r) = h(R_0 \otimes 1_s, G^{(r)}) \quad (13)$$

for an arbitrary function $h$, where $G^{(r)}$ satisfies (3). We hold on to the object-independence of the noise, but drop additivity as a general requirement. Unbiasedness is no longer relevant. It may appear that (3) + (13) are a substantial weakening, jeopardizing the usefulness of the measurement as a $R_0$ measurement. But this is not true. Note first that in the standard approach, relation (6) is invertible (for suitable $G_s^{(r)}$ distributions). We may write this according to

$$\forall \rho_0 \quad P_{R_0}(dr) = \int_{-\infty}^{\infty} P_{R'(r)}(dr') f(dr, r'). \quad (14)$$

The “function” $f(dr, r')$ in (14) is (unlike $g(dr, r')$ in (6)) not necessarily non-negative definite. The inversion (14) is a deconvolution [19]. Via such an inversion all linear functionals $\langle f(R_0) \rangle$ of the $R_0$ probability distribution, $\langle R_0 \rangle$ among them, can be established, regardless of the validity of (4). The more general case of (13) also leads to (6), generalizing (6’) to

$$\forall r \quad g(R, r) = 1,$n
$$\forall r, \Delta r' \quad g(\Delta r', r) \geq 0. \quad (6’’)$$

The smearing is no longer a convolution. Nevertheless the inversion is still often possible [20]. Therefore the expectation value may still be recovered. Even if (6’’) cannot be inverted, its form guarantees that many linear functionals of the $R_0$ distribution can be estimated using this type of noisy measurement. Accordingly, despite differences in emphasis, an approach based on (13) + (3) is not a priori incompatible with expectation-value based approaches since it is often possible to derive $\langle R_0 \rangle$ from the $R'$ results if only (13) is required. On the other hand, it is also quite possible that (13) holds if only (4) is required. But the expectation value $\langle R_0 \rangle$ no longer takes absolute precedence among all $\langle f(R_0) \rangle$. Hence dropping (4) does not imply a weakening of the definition of noisy measurement; on the contrary a definition along the lines of (13) + (3) is a more restrictive one than that used in approaches based exclusively on the expectation value, such as Arthurs and Goodman’s. This difference in outlook on the content of “measurement” may be expressed by saying that we aim at an $R_0$ measurement, whereas expectation-value based approaches aim to make a $\langle R_0 \rangle$ measurement.

For reasons of convenience, we shall take (7)+(6’) (or (6)+(6’’)) as primary, rather than
The interpretation, as object-independent non-additive noise, remains the same, however. This notion, referred to as non-ideality [20], was used earlier by, e.g., Davies [4], Prugovekči [21] and others (cf. also refs. [6,22]). All of these authors also used unbiasedness either implicitly or explicitly, however. Thus all aforementioned objections to unbiasedness apply to these papers. We emphasize, however, that the nature of (7) + (6") makes the requirement of unbiasedness superfluous. In fact the spectra of \( R'(r) \) and \( R_0 \) are allowed to be completely different without embarrassing (7) + (6") indeed without having any effect on the usability of the non-ideal measurement at all. The object-independence (3) of the noise is reflected in Arthurs and Goodman's approach is applicable, (7) + (6") by the fact that the function \( g(\Delta r', r) \) is purely a property of the measuring apparatus. Thus such a smearing represents a type of inaccuracy that is a meter property. Any quantum limitation on such functions in the context of joint measurement of incompatible observables can be seen as a fundamental quantum limit to measurement accuracy. The inequality (9) can be rewritten to show an example of such a limit to the functions \( g(r') \) and \( g(r) \) which describe the non-ideality in the joint measurement of \( R_0 \) and \( T_0 \):

\[
\epsilon_g(r') \epsilon_g(r) \geq \frac{1}{2} |c| , \tag{9'}
\]

where (cf. ref. [22])

\[
\epsilon_g(r) := \left( \int_{-\infty}^{\infty} \tilde{g}(r') r'^2 \, dr' \right)^{1/2}
\]

represents the amount of \( R_0 \) non-ideality, and where the amount of \( T_0 \) non-ideality \( \epsilon_g(r) \) is defined analogously. Here we used (6"). The quantity \( \epsilon_g(r) \) characterizes how much the \( R_0 \) distribution is "smeared" in the device; how much non-ideality is present in the \( R_0 \) measurement. The quantity \( \epsilon_g(r) \) is purely an ancilla property because the function \( g \) is. Seen in this light, the inequality (9') is a relation very different from the usual Heisenberg uncertainty relation. Born's statistical interpretation of quantum theory gives the latter only the meaning of a limit to the amount of scatter in independent accurate measurements [23]; it relates two object properties. Relation (9'), on the other hand, relates meter properties. It may be interpreted as a limit to measurement accuracy, as an uncertainty relation for the measuring instrument, as a relation unconnected to uncertainty relations for the object.

5. Examples

Whether relying on (3) or on (4) restricts the physical applicability most can only be decided by studying concrete examples. Consider first the joint measurement of the spin-\( \frac{1}{2} \) observables \( \sigma_{x,0} \) and \( \sigma_{y,0} \), Yuen's method is not applicable here: \( \sigma_{x,0} + i \sigma_{y,0} \) has only one eigenstate, instead of the four we need. Unbiasedness (4) can therefore certainly not be met.

Arthurs and Goodman's approach is applicable, and leads to the inequality

\[
[\langle \Delta^2 \sigma_{x}'(\tau) \rangle - \langle \sigma_{x,0} \rangle][\langle \Delta^2 \sigma_{y}'(\tau) \rangle - \langle \sigma_{y,0} \rangle] \geq \frac{1}{4} |\langle \sigma_{z,0} \rangle|^2 . \tag{15}
\]

The bound in the above inequality clearly depends on \( \rho_0 \). For certain \( \rho_0 \) it can even be zero. Thus (15) does not give an unambiguous bound to the accuracy with which \( \sigma_{x,0} \) and \( \sigma_{y,0} \) are jointly measurable. This is all the more unsatisfactory, as these spin observables are maximally incompatible [24].

The spin-\( \frac{1}{2} \) case can be treated as a straightforward application of non-ideality: we take \( \sigma'_x = \sigma_{x,0} \otimes \sigma_{x,a} \) and \( \sigma'_y = \sigma_{y,0} \otimes \sigma_{y,a} \) on \( \mathcal{H}_0 \otimes \mathcal{H}_a \). These operators commute, and can thus be measured jointly. Since a measurement of \( \sigma_{x,0} \) has only two possible outcomes, as does a measurement of \( \sigma'_x \), the measurement's quality is characterized by the probability of obtaining the wrong outcome. This error probability is directly related to the \( \sigma_{x,a} \) distribution:

\[
P_{\sigma_x}(\text{wrong}) = P_{\sigma_{x,a}}(1) .
\]

It can then be shown straightforwardly that

\[
[P_{\sigma_x}(\text{wrong}) - \frac{1}{2}]^2 + [P_{\sigma_y}(\text{wrong}) - \frac{1}{2}]^2 \leq \frac{1}{4} , \tag{16}
\]

giving a lower bound for the amount of non-ideality in such a measurement (cf. refs. [20,25]). The bound in inequality (16) is non-trivial for any object state. Hence the more stringent requirements on "noisy measurement" inherent in non-ideality also lead to a more stringent bound to the amount of noise. The spin-\( \frac{1}{2} \) case is mathematically equivalent to certain neutron interferometric experiments. In these ex-
experiments the aim is a joint detection of “path” and “interference”. Complementarity can be shown to appear in these experiments in terms of non-ideality, in accord with the above discussion [26].

The joint measurement of boson number and position furnishes another example for the application of non-ideality. Since the spin-1/2 case bears many formal similarities to the position–momentum case (see refs. [20,25] and references therein), this is the first example of a joint measurement fundamentally different from the position–momentum case. The number operator $N_0$ and the position operator $X_0$ are defined in terms of boson annihilation and creation operators:

$$N_0 = a_0^+ a_0, \quad X_0 = \frac{1}{2} (a_0^+ + a_0).$$

(17)

Consider the following operators on $\mathcal{H}_0 \otimes \mathcal{H}_a$:

$$a_0 := \sqrt{\epsilon} \ a_0 \otimes 1_a + \sqrt{1 - \epsilon} \ 1_b \otimes a_a,$$

$$a_c := \sqrt{1 - \epsilon} \ a_0 \otimes 1_a - \sqrt{\epsilon} \ 1_b \otimes a_a,$$

$$N_0 = a_0^+ a_b, \quad X_c = \frac{1}{2} (a_c^+ + a_c).$$

(18)

The operators $a_0$ and $a_c^+$ (as well as $a_0$ and $a_c$) have all the properties of boson annihilation and creation operators. The operators $N_0$ and $X_c$ have the properties of a number operator and a position operator, respectively. Moreover, since $[a_0, a_c]_+ = 0$ and $[a_b, a_c^+]_+ = 0$, they commute. Now we jointly measure on $\mathcal{H}_0 \otimes \mathcal{H}_a$ the operators $N_0$ and $X_c$. A realization of this procedure is not difficult to envisage: let $0$ represent one mode of the EM field, incident on a beam-splitter with transmittivity $\epsilon$. The outgoing beams are labeled $b$ (transmitted) and $c$ (reflected). We measure in the $b$ beam photon number: $N_b$, and in the $c$ beam “position”: $X_c$ (e.g. by means of homodyning [12]). We further take

$$\rho_0 = \lvert 0 \rangle \langle 0 \rvert,$$

where $\lvert 0 \rangle$ is the harmonic oscillator ground state. Then it can be shown that

$$g(dx, y) = \frac{dx}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2\sigma^2} (x - \sqrt{1 - \epsilon} y)^2\right),$$

$$\sigma = \sqrt{\frac{1}{2} \epsilon},$$

$$\mu_{nm} = 0, \quad \text{if } m < n,$$

$$= \binom{m}{n} \epsilon^n (1 - \epsilon)^{m-n}, \quad \text{otherwise}.$$

Both non-ideality relations (19a) and (19b) are invertible. In the case of (19a) this inversion takes place through deconvolution, analogous to (14), because of

$$X_c = \sqrt{1 - \epsilon} X_0 - \sqrt{\epsilon} X_a.$$

In the case of (19b) we have

$$P_{N_0}(n) = \sum_{m=0}^{\infty} \nu_{nm} P_{N_b}(m),$$

with

$$\nu_{nm} = 0, \quad \text{if } m < n,$$

$$= \binom{m}{n} \epsilon^{-n} (1 - \epsilon)^{m-n}, \quad \text{otherwise}.$$

Note that, contrary to $\mu_{nm}$, the matrix $\nu_{nm}$ contains negative elements (cf. (14)). As $\epsilon \to 0$ the $N_0$ measurement becomes more and more non-ideal, whereas the $X_0$ measurement approaches ideality. If $\epsilon \to 1$ the opposite happens. This shows the complementarity between the measurements of $N_0$ and $X_0$. As noted before, such complementarity in measurement is not reflected in the Heisenberg uncertainty principle at all.

Relation (19b) is equivalent to the measurement of photon number using a detector with quantum efficiency $\epsilon$ [27]. The non-ideality of such an efficient detector can be seen as noise in the $N_0$ measurement. That it can be seen as independent of the object is the reason for the possibility of representing it by a fixed matrix $\mu_{nm}$ independent of $\rho_0$. That $\mu_{nm}$ is not only a function of $n-m$ implies that the noise is non-additive. The invertibility of both marginals makes it possible to estimate the expectation value of both position and number. But in fact all other linear functionals of the probability distributions of the observables we wanted to measure, can also be extracted from the noisy measurement.
For arbitrary pairs of observables on finite dimensional spaces the joint non-ideal measurement is studied elsewhere [25]. As we saw, limits to non-ideality are limits to meter properties. They therefore form a new type of uncertainty principle, an inaccuracy principle. We further applied the non-ideality notion and the inaccuracy principle to quantum non-demolition measurements, e.g. of photon number [28]. Non-ideality has also been studied in connection with the Bell inequalities [29].

6. Conclusions

We have shown that noise constituting measurement inaccuracy can be separated from noise rooted in object fluctuations by considering a characterization of measurement by probability distributions rather than by expectation values. This makes insistence on unbiasedness superfluous. Thus, “non-ideality” can be seen as a definite relation between the distribution of the realized observable and that of the desired observable. This relation may be statistically invertible.

The “non-ideality” concept is, despite its more restrictive definition, widely applicable. This is illustrated by the possibility of applying it to joint measurements of incompatible observables, such as position–momentum, spin-½ quantities and position–number. There it can be seen to lead to complementarity in the respective accuracies, which indeed can be derived quite generally [25]. This type of complementarity embodies a new uncertainty principle, distinct from the Heisenberg uncertainty relations. It allows one to justify statements like “If a measurement of position is made with accuracy \( \Delta q \), and if a measurement of momentum is made simultaneously with accuracy \( \Delta p \), then the product of the two errors can never be smaller than \( \sim h \)” [30]. Such statements, quite in line with Bohr’s thinking, could not be derived using the conventional Heisenberg uncertainty principle, since that only refers to accurate, independent measurements. Approaches that mix the object noise and apparatus noise can also not show such a new principle to exist.

Hence a picture emerges where complementarity is reflected in a dichotomic uncertainty principle: the Heisenberg principle that limits preparation is supplemented by a second uncertainty principle, the inaccuracy principle, that limits measurement. Only both uncertainty principles together form a full quantum theoretical reflection of Bohrian complementarity [31].

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References