Foundations of Physics Letters, Vol. 7, No. 4, 1994

ON THE POSSIBILITY OF MEASURING THE ELECTRON SPIN IN AN INHOMOGENEOUS MAGNETIC FIELD

Hans Martens and Willem M. de Muynck

Department of Theoretical Physics
Eindhoven University of Technology
POB 513, 5600 MB Eindhoven, The Netherlands

Received August 14, 1993; revised January 24 1994

It is widely held that Bohr has shown that the spin of a free electron is not measurable. We point out that Bohr's argument has some important ifs and buts. A concrete configuration is calculated to produce a clear spin separation. This is then shown not to contradict Bohr's reasoning.

Key words: Bohr, electron spin measurement, Stern-Gerlach device.

1. INTRODUCTION

One of the simplest textbook illustrations of the quantum measurement has always been the Stern-Gerlach (SG) device [1,2]. It is therefore perhaps not surprising that, as a byproduct of the recently revived interest in quantum measurements, the SG has again become subject of study [3-5]. In the present paper we shall focus on the issue of the fundamental possibility of the SG for electrons. There is a well-known argument by Bohr that is widely believed to entail that it is impossible to perform such a measurement for free electrons (or protons), because of the size of their magnetic moment with respect to their mass and charge. After a short informal description of the SG, we devote sec. 2 to Bohr's essentially semi-classical argument. A full quantum calculation for a concrete situation is presented in sec. 3, where we find that the electron spin can be measured in principle. Finally, in sec. 4 we discuss the results, and find that there is in fact no contradiction with Bohr's objections.

In the SG the spin of a particle is oriented by means of a strong magnetic field. The direction of this field determines which spin component is measured. An inhomogeneity in the field then causes a spin-dependent force, which will eventually split the wave packet into two packets according to spin value. If the initial direction of the particles' velocity is perpendicular to the magnetic field, the particles are deflected. The deflection can be detected by letting the particles impinge on a screen. If the initial velocity is parallel to the field, the particles are accelerated or decelerated. This type of splitting will involve e.g. differences in arrival time at some detector. The former situation might be referred to as a *transversal* SG, the latter as a *longitudinal* SG.

In both cases, the splitting can be used to *prepare* the particles in a definite spin state. Since the SG process can alternately be viewed as a transfer of spin information to the particle's momentum, we can also use the SG to *measure* a spin component by determining momentum. Accordingly, we shall focus on the SG as a quantum measurement device, in which the spin degrees of freedom play the role of object, whereas the spatial variables represent apparatus observables. For a measurement of spin in the z-direction, the essential term in the Hamiltonian of the "interaction" between "object" and "apparatus" is

$$\hat{H}_{\text{i}} = \beta \hat{z} \hat{\sigma}_z \,, \tag{1}$$

where β is proportional to the strength of the inhomogeneity, and operators are careted. The Pauli spin matrices are denoted by $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ (with eigenvalues $s = \pm 1$). Clearly, a fully consistent description requires other terms as well [2,5]. Still, the most important effect in this context, the correlation between the read-out observable $\hat{p}_z$ and the measured observable $\hat{\sigma}_z$, already appears if we apply (1) alone:

$$\hat{p}_z(t) = \hat{p}_z(0) + \beta t \hat{\sigma}_z \,. \tag{2}$$

In this simplistic SG description, a read-out of $\hat{p}_z$ corresponds to a definite value for $\hat{\sigma}_z$ as soon as the packet separation is larger than the initial momentum spread.

2. BOHR'S ARGUMENT: CLASSICAL TREATMENT

For the above type of reasoning to work, it is essential that the particles are (like silver atoms) neutral. But can it be extended to charged particles, electrons in particular? Bohr studied this problem in depth in the late twenties, and came to the conclusion that the extension to electrons is not possible. His reasoning was never formally published, and only rather sketchy manuscripts exist [6,7]. We give an account of Pauli's version [8], which was presented at the 1930 Solvay conference (cf. also [9]).

Pauli first considers the longitudinal SG. An electron moves in the z-direction, through an inhomogeneous magnetic field. The field's direction lies in the xz-plane, its main component is pointed along the z-axis. Hence the spin-dependent force that is interesting for the measurement is the one in the z-direction:

$$F_{\text{spin},z} = \mu_{\text{B}} \frac{\partial B_z}{\partial z} . \tag{3}$$

The time t_s it takes to decelerate an electron to a standstill is given by

$$m_e v_z = \mu_{\text{B}} \frac{\partial B_z}{\partial z} t_s , \tag{4}$$

after which it turns around. Because $\nabla \cdot \mathbf{B} = 0$ (this condition is crucial in Bohr's reasoning),

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_z}{\partial z} . \tag{5}$$

If we assume the field to be parallel to the z-axis at $x = 0$, the field strength at a distance Δx from the z-axis is therefore given by

$$B_x = -\frac{\partial B_z}{\partial z} \Delta x . \tag{6}$$

This field (through the Lorentz force) achieves a reversal of the velocity v_z within a period

$$t_{\text{c}} = \left| \frac{m_e}{e B_x} \right| . \tag{7}$$

In order for the velocity reversal to be unambiguously attributable to the particle's spin, we must have $t_s \ll t_{\text{c}}$. Using (6), this implies

$$\frac{v_z}{\mu_B} \ll \frac{1}{e \Delta x} \;\Rightarrow\; m_e v_z \Delta x \ll \frac{\hbar}{2} . \tag{8}$$

Inequality (8), although formally different from the uncertainty relation $\Delta x \Delta p_x > \hbar$, still limits the device's operation, Pauli argues. If the beam is made so narrow as to allow for (8), diffraction will come into play, blurring the beam. This means that the wave properties of the electron have to be considered, and that one is outside the realm of classical mechanics.

Next Pauli turns to the transversal SG. The electrons move in the y-direction. Again the spin-dependent force is given by (3). The Lorentz-force in the z-direction is

$$F_{\text{L},z} = -e v_y B_x . \tag{9}$$

Consequently, the magnitude of the variation of this force over the width Δx of the beam is

$$\Delta F_{\mathrm{L},z} = ev_y \left|\frac{\partial B_x}{\partial x}\right| \Delta x = ev_y \left|\frac{\partial B_z}{\partial z}\right| \Delta x \ , \tag{10}$$

again using (5). If the separation is not to be blurred by the variation of the Lorentz-force, clearly we must have

$$\Delta F_{\mathrm{L},z} \ll F_{\mathrm{spin},z} \ \Rightarrow \ v_y \Delta x \ll \mu_{\mathrm{B}}/e = \frac{\hbar}{2m_e} \ , \tag{11}$$

leading to the same conclusion as before. Finally, Pauli discusses some more complicated setups, with yet again the same results.

Prior to arriving at the above reasoning, Bohr believed that the free electron spin could not be detected at all, because it has no classical analog. Like in his other thought experiments, he attempted to base this on an uncertainty relation, in this case $\Delta L \Delta\theta \geq h$. Since $\Delta\theta \leq 2\pi$, ΔL must be larger than $\hbar$, prohibiting any effect from a spin of magnitude $\frac{\hbar}{2}$ to be seen. But such an uncertainty relation does not hold. Later scattering experiments by Mott [10] indeed showed the polarizability of electrons, and Bohr was forced to qualify his reasoning [6,11]. The inequality (8) he ends up with in the argument presented here, involving the product $p_y \Delta x$, does not contradict any uncertainty relation (as Pauli acknowledges). In fact no uncertainty relation at all is used in the argument, and in this sense it differs from most other Bohr *Gedanken* experiments.

Summarizing, Bohr's and Pauli's conclusion is not an absolute interdiction. They hold that the SG cannot work *using the classical path concept.* If the electron is bound, e.g. to an atom, its spin may be detected as a part of the orbital angular momentum. But then the use of the stationary state concept precludes any visualization in terms of an electron trajectory [8,12].

3. QUANTUM TREATMENT: LANDAU STATES

This *caveat* directs us to a possible way out of Bohr's conclusion: a *quantum mechanical* treatment of the SG might give a significant effect after all. Therefore we look again at our setup. We have a strong magnetic field in the z-direction. What Pauli and Bohr ignore is that in such a field the electron's motion in the xy-plane is quantized into Landau orbits [13]. This opens up the possibility of measuring spin by using a coupling of spin to orbital momentum in these Landau states.

First consider the homogeneous field case. We take a cylindrically symmetric gauge in which the vector potential is $\mathbf{A} = B_0(\frac{\alpha}{2}y, -\frac{\alpha}{2}x, 0)$, so that the magnetic field is given by $\mathbf{B} = B_0(0, 0, a)$. The electron Hamiltonian for xy-motion under these circumstances is

$$\hat{H}_0 = \frac{1}{2m_e}\left[\left(\hat{p}_x + \frac{eaB_0}{2}\hat{y}\right)^2 + \left(\hat{p}_y - \frac{eaB_0}{2}\hat{x}\right)^2\right] - \frac{g}{2}aB_0\mu_B\hat{\sigma}_z \,. \tag{12}$$

Choosing natural units ($2m_e = \hbar = e = B_0 = 1$; $\mu_B = \frac{1}{2}e\hbar/m_e = 1$), and taking $g = 2$, eq. (12) can be rewritten as

$$\hat{H}_0 = \hat{p}_x^2 + \frac{a^2}{4}\hat{x}^2 + \hat{p}_y^2 + \frac{a^2}{4}\hat{y}^2 - a(\hat{L}_z + \hat{\sigma}_z) \,. \tag{13}$$

For the moment, we drop the spin part and take $a = 2$. Changing to polar coordinates (r, φ) in position representation, and solving the angular part, gives for the radial part $\phi(r)$ the eigenvalue equation

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \left(\frac{m^2}{r^2} - 2m + r^2\right)\phi = E\phi \,, \tag{14}$$

where m is the eigenvalue of angular momentum $\hat{L}_z$. This equation can be solved in terms of Laguerre polynomials [14], to get

$$\begin{aligned} \phi_{km}(r) &= c_{km}\exp(-\frac{r^2}{2})r^{|m|}L_k^{(|m|)}(r^2) \,; \\ c_{km} &= \left(\frac{k!}{(k+|m|)!}\right)^{\frac{1}{2}} \,; \; k = |m|, |m|+2, |m|+4, \ldots \,; \\ E_{km} &= 2 + 2(k - m) = 2, 6, 10, \ldots \,. \end{aligned} \tag{15}$$

Including the contribution of spin and the magnetic field dependence, the total energy is therefore $E_{kms} = (k - m + 1 - s)a$ (where $s = \pm 1$). We may expect that a small inhomogeneity in the magnetic field will lead to an adiabatic drift in the Landau levels, proportional to E, during which only a negligible amount of transitions between levels take place. If, therefore, the initial spatial part of the state is such that up and down spin do not contribute to the same E-value, the final spatial part can be used to determine initial spin.

In view of its higher degree of symmetry, we shall focus on a longitudinal setup. The electrons initially move in the z-direction, along the major magnetic field direction, and the inhomogeneity is also z-oriented: $a = 2 - bz$, $b \ll 1$. This results in a magnetic field $\mathbf{B} = (\frac{1}{2}bx, \frac{1}{2}by, 2-bz)$. Note that indeed $\nabla\cdot\mathbf{B} = 0$, and that cylindrical symmetry is maintained. The full Hamiltonian now includes some extra terms:

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{p}_z^2 + \hat{H}_{\rm i} \; ; \\ \hat{H}_{\rm i} &= b\hat{z}(\hat{L}_z + \hat{\sigma}_z - \hat{r}^2) - \frac{b}{2}\hat{r}\hat{\sigma}_r + \frac{b^2}{4}\hat{r}^2\hat{z}^2 \\ &= -\frac{b\hat{z}}{2}\hat{H}_0 + \frac{b}{2}[\hat{z}(\hat{p}^2 - \hat{r}^2) - \hat{r}\hat{\sigma}_r + \frac{b}{2}\hat{r}^2\hat{z}^2] \; ; \qquad (16) \\ \hat{\sigma}_r &= \hat{\sigma}_x \cos\phi + \hat{\sigma}_y \sin\phi \; ; \; \hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 \; .\end{aligned}$$

In (16) spin and orbital momentum are coupled; only $\hat{J}_z = \hat{L}_z + \frac{1}{2}\hat{\sigma}_z$ is conserved. The terms responsible for transitions are those between square brackets, the first term in (16) causing the desired force in the z-direction.

In a quantum measurement context, it is natural to require that "object" and "apparatus" are initially uncorrelated. Here that corresponds to demanding that the initial state $\rho_{\rm space+spin}$ be a product of spin and spatial parts:

$$\rho_{\rm space+spin} = |\phi\rangle_{\rm space}\langle\phi| \otimes \rho_{\rm spin} \; . \qquad (17)$$

where $|\phi\rangle_{\rm space}$ is the fixed initial spatial state. After a time t of combined evolution of spin and spatial degrees of freedom, the two become entangled. The probability of then reading out result p_z can be written in terms of an operator-function $\hat{M}(p_z)$ on the spin Hilbert space:

$$\text{prob}_{\hat{\rho}_{\rm spin}}(p_z) = \text{Tr}\,[\rho_{\rm space+spin}(t)\,|p_z\rangle_{\rm space}\langle p_z|] = \text{Tr}[\hat{\rho}_{\rm spin}\,\hat{M}(p_z)] \; , \qquad (18)$$

which can be derived from the initial spatial state and the unitary combined evolution $\hat{U}(t)$ of spatial and spin degrees of freedom:

$$\begin{aligned}\hat{M}(p_z) &= {}_{\rm space}\langle\phi|\hat{U}^\dagger(t)\Big(|p_z\rangle_{\rm space}\langle p_z| \otimes \hat{1}_{\rm spin}\Big)\hat{U}(t)|\phi\rangle_{\rm space} \; ; \qquad (19) \\ \hat{U}(t) &= \exp(-i\hat{H}t) \; .\end{aligned}$$

In the simplistic description (2), the spin-up and spin-down wave packets eventually occupy different p_z-areas. Then to each p_z there corresponds one $\hat{\sigma}_z$ value. In other words, for each p_z $\hat{M}(p_z)$ is proportional to a $\hat{\sigma}_z$ projector. More generally, although $\hat{M}(p_z)$ will still be positive and normalized, we cannot expect it to be proportional to a projector. An object $\{\hat{M}(p_z)\,{\rm d}p_z\}$ with these properties is called a positive operator-valued measure (POVM) [15], generalizing the notion of projection-valued measures.

We read out the p_z values in order to find out something about $\hat{\sigma}_z$. As we observed earlier, we must therefore choose the initial "apparatus" state (i.e. the spatial part) such that only one s value contributes to each E value. In terms of the POVM, this means we want to relate $\hat{M}(p_z)$ to $\hat{\sigma}_z$. Consider first the Hamiltonian (16). It is symmetric under rotations around the z-axis, in particular over an angle π:

$$\hat{S}\hat{H}\hat{S}^\dagger = \hat{H} \;\; ; \;\; \hat{S} = \hat{I}_{xy} \otimes \hat{\sigma}_z \,, \tag{20}$$

$\hat{I}_{xy}$ denoting a rotation of the spatial part over π around the z-axis. We now require that the initial spatial state $|\phi\rangle_{\text{space}}$ be an eigenstate of $\hat{I}_{xy}$:

$$\hat{I}_{xy} \otimes \hat{1}_{\text{spin}} \rho_{\text{space+spin}} = \rho_{\text{space+spin}} \hat{I}_{xy} \otimes \hat{1}_{\text{spin}} = \pm \rho_{\text{space+spin}} \,. \tag{21}$$

This brings about the desired relationship between the POVM and $\hat{\sigma}_z$: as a consequence of (20) and (19), $\{\hat{M}(p_z)\,\mathrm{d}p_z\}$ will be at all times compatible with $\hat{\sigma}_z$ on the spin Hilbert space. Therefore it can be written

$$\hat{M}(p_z) = \sum_{s=\pm 1} f_s(p_z) \hat{P}_s \,. \tag{22}$$

The f_s are positive functions and $\hat{P}_s$ denote the projectors onto the $\hat{\sigma}_z$ eigenstates. A spin value s will lead to outcome p_z with probability $f_s(p_z)$; a measurement outcome p_z corresponds to spin value s with likelihood $f_s(p_z)$. In general the functions f_s overlap to some extent. Then there clearly is a finite probability of deducing the wrong spin value from the p_z read-out: the measurement is non-ideal [16]. A convenient measure for the the measurement quality is the indistinguishability u of $f_{\pm 1}$, derived from (22) via

$$u = \int_{-\infty}^{\infty} [f_{+1}(p_z) f_{-1}(p_z)]^{\frac{1}{2}} \,\mathrm{d}p_z \,. \tag{23}$$

Ideally, when the separation according to spin is perfect, the f's do not overlap, and $u = 0$. But initially there is no separation so that then $u = 1$. As time proceeds, the spin-up and spin-down packets will separate, the measurement gets better and u will decrease to a certain limit. The limit value depends on the strength of the field, compared to the size of the inhomogeneity [5]. The reason for this is that the direction of the spin precession axis varies as a function of r due to the field inhomogeneity. The stronger the field is, the smaller these variations are, and the lower the achievable measurement error is.

In the transversal configuration, the symmetry (20) only holds if an electric field in the x-direction would compensate the Lorentz-force

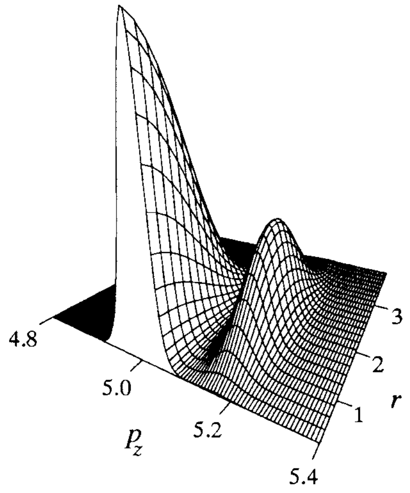

Figure 1: Result of a typical calculation. We plotted $\frac{1}{r}\mathrm{prob}(\mathrm{p_z}, \mathrm{r})$. (Data: $bt = 0.1$; $b = 0.0003$. The initial z-state was Gaussian, centered around $p_z = 5$ with standard deviation $\delta_z = 15$; the initial xy-state a $k = m = 0$ Landau state; the initial spin-state a $\sigma_x = 1$ eigenstate.)

due to p_y [5]. But if p_y is not sharp, this method works approximately at best. In the longitudinal setup, (22) holds exactly irrespective of p_z, or of the value of b. Moreover, the longitudinal setup's cylindrical symmetry considerably simplifies the device's analysis [17].

As an illustration of this general argument we integrated the Schrödinger equation numerically on the basis of (16) and (15). For the spatial part $|\phi\rangle_{\mathrm{space}}$ of the initial state we took a cylindrical symmetric Gaussian: the $k = m = 0$ Landau ground state. In Fig. 1 we plotted

$$\frac{1}{r}\mathrm{prob}(p_z, r) = \mathrm{Tr}\left[\rho_{\mathrm{space+spin}}(t)\,|p_z, r\rangle_{\mathrm{space}}\langle p_z, r|\right] \qquad (24)$$

of a typical calculated state. (This spatial distribution is cylindrically symmetric, so we omitted its φ-dependence.) Note that because the drift is proportional to E rather than s, the acceleration of the packets is not symmetrical. The spin-up packet, with $E = 0$, is not accelerated and has roughly retained its shape; the spin-down packet, with $E = 2a$, has become ring-shaped. Nevertheless, the packets are very well separated with regard to p_z.

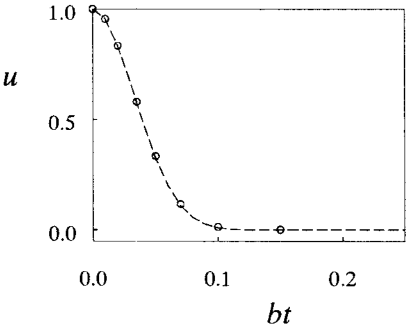

Figure 2: The measurement's p_z-indistinguishability u as a function of time. The results of the numerical calculation are indicated by circles. Initial data as in Fig. 1. The u-curve for two linearly separating Gaussian packets, $u(t) = \exp(-(2bt\delta_z)^2)$, is included (dashed line).

In Fig. 2 we plotted the indistinguishability u as a function of time. Indeed u, and therefore the measurement error, becomes quite small. In fact, despite the change of shape the packets have undergone, the indistinguishability is seen to agree with that of the simplest model for the SG, two Gaussian packets splitting according to (2), which is also plotted in Fig. 2.

4. CONCLUSIONS

The good measurement quality and clear packet separation we achieved may appear to contradict Bohr's conclusion. But this is in fact not so. We must have clearly defined Landau levels, and the condition $b \ll 1$ implies that the time necessary for clear separation is much larger than the orbit time. Bohr's point in section 2 that the use of stationary states precludes the use of a classical trajectory (although he undoubtedly had atoms in mind) is valid here, too. In the presence of a non-negligible magnetic field, the electrons are not really free so that a classical trajectory is in general not a usable concept.

From the point of view of feasibility, the constraints (17) and (21) on the initial state are probably difficult to fulfill, especially in view of the small length scales involved (of the order $\hbar^{\frac{1}{2}}(eB_0)^{-\frac{1}{2}}$). But nevertheless we can conclude that there is no *fundamental* reason why electron spin measurement by means of a SG should in quantum theory be impossible in principle.

REFERENCES

1. R. Feynman, R. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. 3 (Addison-Wesley, Reading, Mass., 1965).
2. G. Ludwig, *Einführung in die Grundlagen der Theoretischen Physik*, Vol. 3 (Vieweg, Braunschweig, 1976).
3. R. Levine and R. Tucci, *Found. Phys.* **19**, 175 (1990).
4. M. Scully, B.-G. Englert and J. Schwinger, *Phys. Rev. A* **40**, 1775 (1989).
5. H. Martens and W. de Muynck, *J. Phys. A* **26**, 2001 (1993).
6. N. Bohr, *Collected Works*, Vol. 6, J. Kalckar, ed. (North Holland, Amsterdam, 1985), pp. 305-350.
7. N. Bohr, unpublished manuscript (1929), reprinted on p. 333 of [6].
8. W. Pauli, in *Le Magnétisme*, proceedings of the 6th Solvay conference (Gauthier-Villars, Paris, 1932), p. 217, reprinted on p. 339 of [6].
9. N. Mott & H. Massey, *The Theory of Atomic Collisions* (Clarendon, Oxford, 1965).
10. N. Mott, *Proc. R. Soc. A* **124**, 425 (1929).
11. O. Darrigol, *Historical Studies in the Physical Sciences* **15** (1), 38 (1984).
12. N. Bohr, *Atomic Theory and the Description of Nature* (Cambridge University Press, Cambridge, 1934), Introduction, reprinted on p. 257 of [6].
13. L. Landau, *Z. Phys.* **64**, 629 (1930).
14. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
15. C. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
16. H. Martens and W. de Muynck, *Found. Phys.* **20**, 255 (1993).
17. In the above discussion, we focused on measurement. If preparation of particles with definite spin is our aim, (17) is not warranted. But the spatial symmetry requirement (21) can be seen to be sufficient to establish a clear-cut correlation between $\hat{\sigma}_z$ and $\hat{p}_z$ anyway. Hence (17) is also not needed for preparation.