ON AN ALTERNATIVE INTERPRETATION OF THE BELL INEQUALITIES

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An alternative interpretation is discussed on which the Bell inequalities are not consequences of local realism and on which Einstein–Podolsky–Rosen-like experiments are no tests of the inequalities, because the Bell inequalities only need to be satisfied if all observables are measured jointly.

The violation of the Bell inequalities by quantum mechanics is generally interpreted as a demonstration of the impossibility to reproduce all results of quantum mechanics by means of a local realistic (hidden variables) theory. Moreover, the discrepancy between quantum mechanics and local realistic theories is thought to be liable to an experimental test by means of Einstein–Podolsky–Rosen-like experiments. The general agreement of these experiments with quantum mechanics is widely accepted as a proof of nonlocality. In this note an alternative interpretation of the Bell inequalities is discussed, or which (i) the Bell inequalities are not a consequence of local realism but follow from the assumption that a joint measurement procedure exists for the four observables that are involved, and (ii) EPR-like experiments are not tests of the Bell inequalities because in such measurements only *two* observables are measured jointly.

We will start our considerations from the point of view that a derivation of the Bell inequalities does not need a recourse to realism. Indeed, as was demonstrated by Fine [1] by deriving the inequalities (in the generalized form deployed by Clauser and Horne [2]) from the mere existence of a joint probability distribution of the four observables that are involved, the Bell inequalities have a meaning if we restrict ourselves to the phenomenalistic level of measurement results. Thus, if $p(a_i^1, a_j^2, a_k^3, a_l^4)$ is the joint probability distribution of the measurement results of observables $\mathscr{A}^1$, $\mathscr{A}^2$, $\mathscr{A}^3$ and $\mathscr{A}^4$, and

$$p(a_i^1, a_j^2) = \sum_{k,l} p(a_i^1, a_j^2, a_k^3, a_l^4) ,$$

etc. are marginals of this jpd, it is straightforward to obtain the Bell/CH-inequalities

$$-1 \leqslant p(a_i^1, a_j^2) + p(a_i^1, a_k^3) + p(a_k^3, a_l^4) - p(a_j^2, a_l^4) - p(a_i^1) - p(a_l^3) \leqslant 0 , \tag{1}$$

without imposing any additional requirement.

Since $p(a_i^1, a_j^2, a_k^3, a_l^4)$ is a positive linear functional on the Hilbert–Schmidt space of bounded linear operators, by the Riesz representation theorem there exists a positive operator R_{ijkl} satisfying

$$p(a_i^1, a_j^2, a_k^3, a_l^4) = \mathrm{Tr}\, \rho R_{ijkl} , \tag{2}$$

$$R_{ijkl} \geqslant 0 . \tag{3}$$

The operators R_{ijkl}, also obeying

$$\sum_{ijkl} R_{ijkl} = I , \tag{4}$$

define a positive operator valued measure. Denoting $\Sigma_{kl} R_{ijkl} = R_{ij}^{12}$, etc., the inequality (1) can be expressed according to

$$-1 \leqslant R_{ij}^{12} + R_{ik}^{13} + R_{kl}^{34} - R_{jl}^{24} - R_i^1 - R_l^3 \leqslant 0 . \tag{5}$$

Note that in this derivation the observables ^{1}, ^{2}, $\mathscr{A}^3$ and $\mathscr{A}^4$ are allowed to be incompatible.

On the purely phenomenalistic level the existence of a joint probability distribution of several observables is equivalent to the existence of a joint measurement procedure of these observables. Hence, it is possible to interpret the inequalities (5) as a consequence of the existence of a joint measurement procedure for the four observables that are involved. If this interpretation is correct, then the EPR-type experiments do not constitute a test of the Bell inequalities because only two of the four observables are measured jointly in these experiments. In order to relate such experiments to the Bell inequalities we would have to construct a jpd of the four observables that are involved, from the probability distributions of the pairs of observables measured in the EPR-experiments. Even if such a construction is possible mathematically [3] the physical significance of merging the outcomes of *different* experiments into one single probability statement is completely obscure in a theory dealing only with measurement results. In *orthodox* quantum mechanics the joint measurement of incompatible observables is held to be impossible. Jpd's of incompatible observables are judged to be meaningless. For this reason the Bell inequalities cannot be derived in this theory, as necessary properties to be obeyed by *all* EPR-experiments.

The apparent disagreement of the Bell inequalities and quantum mechanics stems from the requirement that the probabilities $p(a_i^1, a_j^2)$ etc. in (1) should equal the probabilities which are measured in the EPR-experiments, and which are yielded by orthodox quantum mechanics. Thus, if $A^m = \Sigma_i a_i^m P_i^m$, $m = 1, 2, 3, 4$, we should have

$$p(a_i^m, a_j^n) = \mathrm{Tr}\, \rho P_i^m P_j^n \tag{6}$$

for the combinations (m, n) figuring in (1), which are chosen so as to correspond to *commuting* pairs of operators (i.e. $[A^1, A^2]_- = [A^1, A^3]_- = [A^4, A^2]_- = [A^4, A^3]_- = 0$). Then, for these combinations $R_{ij}^{mn} = P_i^m P_j^n$, which, if inserted in (5), leads to contradictions. Since the Bell inequalities are necessary conditions for the existence of a jpd of the four observables $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$ and $\mathcal{A}^4$, we may conclude that such a jpd cannot exist if the inequalities are violated. This can be interpreted as a demonstration of the impossibility to obtain the results (6) of EPR-experiments from one measurement arrangement in which the observables $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$ and $\mathcal{A}^4$ are measured jointly. This conclusion can be arrived at in a more direct way from an extension of quantum mechanics in which allowance is made for the possibility of a joint measurement of incompatible observables, and, consequently, of jpd's of such observables.

Such an extension was discussed in ref. [4] for the case of two incompatible observables. It was demonstrated that, if it is required that one of the marginal distributions of the jpd equals the single distribution demanded by orthodox quantum mechanics, this entails commutativity of the operators representing these observables. Stated differently, if two *incompatible* observables are measured jointly, the measurements necessarily disturb each other. It is easy to extend this theory to the case of more than two observables measured jointly. If, for instance, on measuring jointly the observables $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$ and $\mathcal{A}^4$, the measurement of $\mathcal{A}^1$ is undisturbed, this means that

$$\sum_{jkl} R_{ijkl} = P_i^1 \ .$$

Applying the derivation, leading to eq. (30) of ref. [4], to R_{ij}^{12}, it follows that $[A^1, A^2]_- = 0$. Analogous derivations show that $[A^i, A^j]_- = 0$ for all $i, j = 1, 2, 3, 4$, if it is required that each of the A^i measurements is undisturbed. Since the assumption (6) precisely has this latter consequence, our theory of joint measurement tells us that all of the four jointly measured observables should be mutually compatible. If the observables are not all compatible, as is the case in the EPR-experiments which are usually considered (for which $[A^1, A^4]_- \neq 0$ and $[A^2, A^3]_- \neq 0$), then the requirements (6) cannot be maintained for the joint measurement because these imply non-disturbance of the marginal probabilities of the single observables. The Bell/CH-inequalities (1), however, being a direct consequence of the existence of the jpd of $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$ and $\mathcal{A}^4$, remain valid in extended quantum mechanics, independently of whether the observables are compatible or not. Since no joint measurements of the four observables have been performed up till now, on this interpretation the Bell inequalities have never been tested experimentally.

Since the existence of the joint probability distribution $p(a_i^1, a_j^2, a_k^3, a_l^4)$ is the only assumption in deriving the inequality (1), neither the issue of realism, nor that of locality is involved. Hence, if orthodox quantum mechanics does not satisfy the inequalities, this cannot be blamed on a failure of reality to be local. Only the properties of the joint probability distributions of quantum mechanics can be held responsible. Clearly, it is the requirement (6) of the equality of the marginal probabilities $p(a_i^m, a_j^n)$ to the experimental probabilities of the EPR-experiments that causes a violation of the Bell inequalities. Now $\mathrm{Tr}\, \rho\, P_i^1 P_j^2$ is the joint probability distribution of the joint measurement of the compatible observables $\mathscr{A}^1$ and $\mathscr{A}^2$, the measurement arrangements for $\mathscr{A}^3$ and $\mathscr{A}^4$ being *absent*. Hence, the equality (6) can be interpreted as a requirement that the marginal probabilities $p(a_i^m, a_j^n)$ of two compatible observables $\mathscr{A}^m$ and $\mathscr{A}^n$, measured jointly with two other (incompatible) observables, are not influenced by the presence of the measurement arrangement for measuring these other observables. Joint probability distributions having this property can be attributed to the object system as properties of this system which are independent of the way they are measured. For this reason, theories in which the probability distributions have this property may be called objective [2] or objectivistic [5]. It is then clear that it is the assumption of objectivism that causes quantum mechanics to violate the Bell inequalities. This assumption is unwarranted both in orthodox quantum mechanics (in which joint probability distributions of incompatible observables do not exist) and in extended quantum mechanics (where the joint probability distributions *do* depend on the measurement arrangement). Quantum mechanics does not violate the Bell inequalities if it is interpreted in a proper nonobjectivistic way, in which the essential role of the measurement process is duly acknowledged.

In the foregoing we did not leave the purely phenomenalistic level of the measurement results. Since the idea of nonlocality has its historical origin in the attempt to reproduce quantum mechanics by means of a realistic theory, it is worthwhile to consider also this realistic level. We will demonstrate that also here the Bell inequalities are essentially derived from the existence of a joint probability distribution of the four observables that are involved. Moreover the assumption of locality will be seen to be an unnecessary additional assumption.

Let us consider first the measurement of one single observable $\mathscr{A}^m$. Then, the probability distribution $p_{\mathscr{A}^m}(a_i^m)$ can be represented as an average over the phase space Λ,

$$p_{\mathscr{A}^m}(a_i^m) = \int_\Lambda \mathrm{d}\lambda \rho(\lambda)\, p_{\mathscr{A}^m}(a_i^m|\lambda)\,, \tag{7}$$

in which $p_{\mathscr{A}^m}(a_i^m|\lambda)$ is the conditional probability to obtain the measurement results a_i^m for $\mathscr{A}^m$ if the hidden variable has the value λ. Hence,

$$\sum_i p_{\mathscr{A}^m}(a_i^m|\lambda) = 1\,. \tag{8}$$

It should be stressed here that no assumption of locality underlies the expression (7). The interaction between object system and measuring instrument may be either local or nonlocal. It is assumed that (7) obtains for each of the four observables involved in the Bell inequalities, each measured singly.

A direct generalization of (7) to the *joint* measurement of the four observables yields the joint probability distribution

$$p_{\mathscr{A}^1, \mathscr{A}^2, \mathscr{A}^3, \mathscr{A}^4}(a_i^1, a_j^2, a_k^3, a_l^4) = \int_\Lambda \mathrm{d}\lambda \rho(\lambda)\, p_{\mathscr{A}^1, \mathscr{A}^2, \mathscr{A}^3, \mathscr{A}^4}(a_i^1, a_j^2, a_k^3, a_l^4|\lambda)\,. \tag{9}$$

Like in quantum mechanics, the Bell inequalities can be derived from this quantity by Fine's method. Since the conditional joint probabilities

$$p_{\mathscr{A}^1, \mathscr{A}^2, \mathscr{A}^3, \mathscr{A}^4}(a_i^1, a_j^2, a_k^3, a_l^4|\lambda)$$

in general need not have the objectivity property discussed above, no conflict with (extended) quantum mechanics has to be expected.

Contrary to quantum mechanics, in a realistic theory the Bell inequalities can be derived from a consideration of *single* measurements. This is so, because from the expressions (7) for $m = 1, 2, 3, 4$ we can formally compose the joint probability distribution

$$p'(a_i^1, a_j^2, a_k^3, a_l^4) = \int_\Lambda \mathrm{d}\lambda_\rho(\lambda)\, p_{\mathscr{A}^1}|\lambda) \times p_{\mathscr{A}^2}(a_j^2|\lambda)\, p_{\mathscr{A}^3}(a_k^3|\lambda)\, p_{\mathscr{A}^4}(a_l^4|\lambda)\,, \tag{10}$$

which has the single probabilities $p_{\mathscr{A}^m}(a_i^m)$ as marginals. Evidently if (7) holds for each of the four observables, then it is possible to imagine a joint measurement procedure for the four observables yielding simultaneously the results of the single measurements. It is evident that this imaginary joint measurement procedure has the objectivity property. Hence our assumptions entail conflict with quantum mechanics.

Since (7) is the only assumption made in order to arrive at this conclusion [the composition of the joint probability distribution (10) just being a formal manipulation that can always be performed if (7) holds for $m = 1, 2, 3, 4$], there seem to remain only two possibilities in order to escape conflict with quantum mechanics: either the expression (7) itself is incorrect, or the simultaneous assumption of (7) for the measurement of different (incompatible) observables is not warranted. It is straightforward to show that (7) generally holds if the interaction between object and measuring instrument is governed by a stochastic operator. For this reason we hold the second possibility to be the suspect one. In the domain of quantum mechanics it does not seem allowed to represent the object, in the context of different measurements, by one and the same probability distribution $\rho(\lambda)$ on one and the same phase space Λ. This means that in the domain of quantum mechanics the object cannot be thought of as *prepared* independently of the presence of the measurement arrangement. Since this can be viewed as an alternative formulation of the nonobjectivity of quantum mechanics, we arrive at the same conclusion as was obtained on the purely phenomenalistic level: the violation of the Bell inequalities is due to an unwarranted assumption of objectivity. In a nonobjectivistic realistic theory no problems arise because here it is taken into account that the preparation of the object system cannot be considered as independent of the presence of the measurement arrangement. As on the phenomenalistic level this implies that the Bell inequalities can only be derived in the context of a joint measurement of the *four* observables that are involved, because only in this situation we have eq. (9) with an unambiguous density $\rho(\lambda)$. If, however, the preparation is not independent of the measurement arrangement, then $\rho(\lambda)$ cannot be chosen the same in (7) for all of the four different measurements. This prevents the construction of the jpd (10) of a formal joint measurement procedure of the four observables. If the issue of (non) objectivity is taken into account the acceptance of a realistic level underneath the quantum mechanical phenomena clearly does not alter the role played by the EPR-experiments in testing the Bell inequalities.

It should be stressed here that in deriving a possible controversy between realistic theories and quantum mechanics we did not consider EPR-like experiments. Hence, the issue of the presence or absence of a nonlocal influence of one measuring instrument on the other does not play any role. Our derivation also applies to such experiments, however. In a local realistic theory the joint probability distributions $p(a_i^m, a_j^n)$ (6) are represented, for the compatible pairs of observables $\mathscr{A}^m$ and $\mathscr{A}^n$, by the expression [2]

$$p_{\mathscr{A}^m,\mathscr{A}^n}(a_i^m, a_j^n) = \int_\Lambda d\lambda\, \rho(\lambda)\, p_{\mathscr{A}^m}(a_i^m|\lambda)\, p_{\mathscr{A}^n}(a_j^n|\lambda)\,. \tag{11}$$

Here it is assumed that locality implies factorization of the conditional probabilities according to

$$p_{\mathscr{A}^m,\mathscr{A}^n}(a_i^m, a_j^n|\lambda) = p_{\mathscr{A}^m}(a_i^m|\lambda)\, p_{\mathscr{A}^n}(a_j^n|\lambda)\,. \tag{12}$$

Since the probabilities (11) are also marginals of the same formal jpd (10), it is evident that the Bell inequalities should hold also in this case. It should be stressed, however, that, apart from locality, in the derivation of these inequalities from (11) *also* objectivity is presupposed. Since, as we saw, the Bell inequalities can be derived from objectivity *alone*, the assumption (12) of locality is clearly inessential. This becomes particularly evident if the derivation is applied to the four observables $\mathscr{B}^1, \mathscr{B}^2, \mathscr{B}^3$ and $\mathscr{B}^4$, representing the four EPR-measurements corresponding to the operators A^1A^2, A^1A^3, A^4A^2 and A^4A^3, respectively. Since a measurement result of A^mA^n consists of a pair (a_i^m, a_j^n), the expressions (7) become

$$p_{\mathscr{B}^1}(a_i^1, a_j^2) = \int_\Lambda d\lambda\, \rho(\lambda)\, p_{\mathscr{B}^1}(a_i^1, a_j^2|\lambda)\,, \tag{13}$$

and analogously for $m = 2, 3, 4$. We can now derive a conflict with orthodox quantum mechanics, *without imposing the locality condition* (12), by considering the formal jpd

$$p'_{\mathscr{B}^1,\mathscr{B}^2,\mathscr{B}^3,\mathscr{B}^4}((a_i^1, a_j^2), (a_k^1, a_l^3), (a_p^4, a_q^2), (a_r^4, a_s^3)) = \int_\Lambda d\lambda\, \rho(\lambda)\, p_{\mathscr{B}^1}(a_i^1, a_j^2|\lambda)\, p_{\mathscr{B}^2}(a_k^1, a_l^3|\lambda) \times p_{\mathscr{B}^3}(a_p^4, a_q^2|\lambda)\, p_{\mathscr{B}^4}(a_r^4, a_s^3|\lambda)\,, \tag{14}$$

corresponding to a formal composition of the results of the four EPR-experiments. From the requirement (6), imposed on the marginals (13) of (14), it follows once more that the observables $\mathcal{B}^m$ should all be mutually compatible, that is $[A^1A^2, A^1A^3]_- = 0$, etc. Taking into account the commutativity of the operators A^m measured jointly, this implies $(A^1)^2[A^2, A^3]_- = 0$. If A^1 represents a spin component, then $(A^1)^2$ is proportional to the unit operator and $[A^2, A^3]_- = 0$, contrary to our assumption.

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