THE BELL INEQUALITIES AND THEIR IRRELEVANCE TO THE PROBLEM OF LOCALITY IN QUANTUM MECHANICS

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A straightforward derivation of the Bell inequalities is given, without in any way appealing to locality. This demonstrates the incompatibility of both local and nonlocal hidden variables theories with quantum mechanics, and the irrelevance of the Bell inequalities to the problem of (non)locality in such theories.

In this note a derivation of the Bell inequalities is given which is not completely new. Essentially, it can be gathered from articles by Fine [1] and by Clauser and Horne [2]. It seems, however, worthwhile to write down the complete derivation once again in order to demonstrate explicitly that the assumptions of locality [2] or factorizability [1] which are made in the cited articles, are not necessary for a proof of the Bell inequalities. In his original derivation Bell [3] assumed his hidden variables theory to satisfy a locality condition which he deemed to be a "vital assumption". Presumably due to this fact there still exists a widespread belief – also among specialists – that the Bell inequalities can *not* be derived for *non*local hidden variables theories. This would leave open the possibility that quantum mechanics might be reproduced by a nonlocal hidden variables theory. From the following, however, it should be clear that the mere existence of hidden variables is sufficient to yield the Bell inequalities. Hence not only local but also nonlocal hidden variables theories are incompatible with quantum mechanics. Local and nonlocal theories being on an equal footing it also follows that the Bell inequalities are completely irrelevant to the problem of (non)-locality in hidden variables theories.

Let A^i, $i = 1,\ldots, n$ be random variables with values in $\mathbb{R}$. We consider a classical probability theory of such random variables in the sense of Kolmogorov's theory of probability. Then, if S^i is a subset of $\mathbb{R}$, the only assumption that will be made is about the existence of joint probabilities $P(\mathrm{S}^1,\ldots,\mathrm{S}^n)$ of random variables A^i taking values in subsets S^i, $i = 1,\ldots, n$. Because of their theoretical meaning the probabilities $P(\mathrm{S}^1,\ldots,\mathrm{S}^n)$ satisfy the following properties:

$$P(\mathrm{S}^1,\ldots,\mathrm{S}^n) \geqslant 0, \tag{1}$$

$$P(\mathrm{T}^1,\mathrm{S}^2,\ldots,\mathrm{S}^n) \leqslant P(\mathrm{S}^1,\mathrm{S}^2,\ldots,\mathrm{S}^n) \quad \text{if } \mathrm{T}^1 \subseteq \mathrm{S}^1, \tag{2}$$

$$P(\mathrm{S}^1,\mathrm{S}^2,\ldots,\mathrm{S}^n) + P(\bar{\mathrm{S}}^1,\mathrm{S}^2,\ldots,\mathrm{S}^n) = P(\mathrm{S}^2,\ldots,\mathrm{S}^n), \quad \text{if } \bar{\mathrm{S}}^1 \text{ is the complement of } \mathrm{S}^1 \text{ in } \mathbb{R}. \tag{3}$$

These properties are sufficient in order to derive the Bell inequalities. To this end I first reproduce the following theorem due to Fine [1],

Fine's theorem:

If A^1, A^2, A^3 and A^4 are arbitrary random variables, then

$$-1 \leqslant P(\mathrm{S}^1,\mathrm{S}^2) + P(\mathrm{S}^1,\mathrm{S}^3) + P(\mathrm{S}^3,\mathrm{S}^4) - P(\mathrm{S}^2,\mathrm{S}^4) - P(\mathrm{S}^1) - P(\mathrm{S}^3) \leqslant 0. \tag{4}$$

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Proof:

$$P(S^1,S^2,S^3)=P(S^1,S^2,S^3,S^4)+P(S^1,S^2,S^3,\bar{S}^4)$$
$$\leqslant P(S^2,S^4)+P(S^3,\bar{S}^4)$$
$$=P(S^2,S^4)+P(S^3)-P(S^3,S^4), \qquad (5)$$

$$P(\bar{S}^1,S^2,S^3)=P(\bar{S}^1,S^2,S^3,S^4)+P(\bar{S}^1,S^2,S^3,\bar{S}^4)$$
$$\leqslant P(S^3,S^4)+P(S^2,\bar{S}^4)$$
$$=P(S^3,S^4)+P(S^2)-P(S^2,S^4), \qquad (6)$$

$$0\leqslant P(S^1,\bar{S}^2,\bar{S}^3)=P(S^1,\bar{S}^2)-P(S^1,\bar{S}^2,S^3)$$
$$=P(S^1)-P(S^1,S^2)-P(S^1,S^3)+P(S^1,S^2,S^3). \qquad (7)$$

From (7) and the equality

$$-P(S^2,S^3)+P(\bar{S}^2,\bar{S}^3)=1-P(S^2)-P(S^3),$$

it follows that

$$0\leqslant P(\bar{S}^1,\bar{S}^2,\bar{S}^3)=P(\bar{S}^2,\bar{S}^3)-P(S^1,\bar{S}^2,\bar{S}^3)$$
$$=1-P(S^1)-P(S^2)-P(S^3)+P(S^1,S^2)$$
$$+P(S^1,S^3)+P(\bar{S}^1,S^2,S^3). \qquad (8)$$

Combining (5) and (7) we get

$$0\leqslant P(S^1)-P(S^1,S^2)-P(S^1,S^3)+P(S^2,S^4)$$
$$+P(S^3)-P(S^3,S^4). \qquad (9)$$

From (6) and (8) it follows that

$$0\leqslant 1-P(S^1)-P(S^3)+P(S^1,S^2)+P(S^1,S^3)$$
$$+P(S^3,S^4)-P(S^2,S^4). \qquad (10)$$

The inequalities (9) and (10) can be combined to, what are called by Fine, the Bell–Clauser–Horne inequalities

$$-1\leqslant P(S^1,S^2)+P(S^1,S^3)+P(S^3,S^4)-P(S^2,S^4)$$
$$-P(S^1)-P(S^3)\leqslant 0. \qquad (11)$$

Analogous expressions can be obtained for other choices of the random variables. For instance, by interchanging the indices 2 and 3 we get

$$-1\leqslant P(S^1,S^2)+P(S^1,S^3)+P(S^2,S^4)-P(S^3,S^4)$$
$$-P(S^1)-P(S^2)\leqslant 0. \qquad (12)$$

This concludes the proof of Fine's theorem.

From (11) and (12) it is straightforward to derive the inequalities as originally obtained by Bell [3] for dichotomic observables with possible values +1 and −1. For such observables the expectation value of the product A^iA^j is given by

$$\langle A^iA^j\rangle=P_{ij}(+,+)+P_{ij}(-,-)-P_{ij}(+,-)-P_{ij}(-,+), \qquad (13)$$

where $P_{ij}(+,+)$ stands for $P(A^i=+1, A^j=+1)$, etc. Now the inequality (11) can be transformed in a straightforward manner into an inequality for the expectations $\langle A^iA^j\rangle$, since by (13) the middle term of (11) can be related to these expectations. Denoting the middle term of (11) by $Q(S^1,S^2,S^3,S^4)$ and using the notation of (13) it can be seen directly that

$$\langle A^1A^2\rangle+\langle A^1A^3\rangle+\langle A^3A^4\rangle-\langle A^2A^4\rangle$$
$$=Q(+,+,+,+)+Q(-,-,-,-)$$
$$-Q(+,-,-,+)-Q(-,+,+,-). \qquad (14)$$

From the inequality (11) it follows that

$$Q(+,+,+,+)\geqslant -1, \quad Q(-,-,-,-)\geqslant -1,$$
$$-Q(+,-,-,+)\geqslant 0, \quad -Q(-,+,+,-)\geqslant 0.$$

Hence

$$(\langle A^1A^2\rangle+\langle A^1A^3\rangle+\langle A^3A^4\rangle-\langle A^2A^4\rangle)\geqslant -2. \qquad (15)$$

From (12) we can derive in a completely analogous way that also

$$(\langle A^1A^2\rangle+\langle A^1A^3\rangle+\langle A^2A^4\rangle-\langle A^3A^4\rangle)\geqslant -2. \qquad (16)$$

From (15) and (16) we finally get

$$|\langle A^2A^4\rangle-\langle A^3A^4\rangle|\leqslant 2+\langle A^1A^2\rangle+\langle A^1A^3\rangle, \qquad (17)$$

which coincides with eq. (15) of ref. [3], if $\langle A^1A^2\rangle = -1$, as it is assumed there.

It should be stressed once more that in the derivation of (17) no special presuppositions have been made regarding the nature of the quantities A^i exceeding the specification of their character as random variables of a classical probability theory with possible values +1 and −1. In particular, there is no assumption of locality. Hence, we may conclude that in Bell's original derivation this assumption is superfluous, even though it was Bell's explicit intention to exhibit an incompatibility between quantum mechanics and *local* hidden variables theories. As a matter of fact, the nonlocality of Bohm's quantum potential [4] is considered by Bell as strong evidence against locality [5]. It is interesting to notice, however, that Bohm [6] does not exclude the possibility of a reconciliation of his nonlocal quantum potential with an underlying local field theory.

Fine [1] explicitly expresses his doubts regarding the role of locality played in the derivation of the Bell inequalities. Yet, he does not seem to contribute very much to a clarification of the issue by introducing his concept of factorizability and by proving the existence of a factorizable model in case the Bell–Clauser–Horne equalities hold. This would imply that to any nonlocal theory giving rise to the existence of joint probabilities $P(S^1, S^2, S^3, S^4)$ there would correspond an equivalent factorizable model having the same joint probabilities. To the criticism by Garg and Mermin [7] regarding the physical relevance of such a model we may add the remark that the existence of the factorizable model does not cause the equivalent nonlocal model to be compatible with quantum mechanics.

I conclude with two remarks. In the first place, the derivation given above should not be interpreted as exhibiting the impossibility of all kinds of realistic (hidden variables) theories. As remarked by Bohm and Hiley [6], the Bell inequalities are derived under the assumption that the probability distribution of hidden variables does not depend on the states of any large scale systems (such as the various pieces of observing apparatus). This point was also raised by Lochak [8]. In this view the Bell inequalities cannot be derived if the probability distribution of the hidden variables is changed by changing the measurement arrangement. An alternative interpretation of the Bell inequalities, in which the role of the measuring instrument is taken into account, is developed by de Muynck and co-workers [9].

The second remark has to do with the problem of (non)locality of the microscopic world as described by quantum mechanics. Of course, this problem is not settled by demonstrating the irrelevance of the Bell inequalities in this matter. From the conclusions of this note it follows, however, that this question, which has an interest for its own sake, must be studied by methods which are independent of the Bell inequalities. To this end, on the level of quantum mechanics a theory of local observables and local operations was developed (de Muynck [10]), demonstrating some nonlocal features. However, whether this nonlocality stems either from certain peculiarities of the quantum mechanical description, or is caused by a veritable nonlocality of an underlying reality remains an open question.

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