ON THE JOINT MEASUREMENT OF INCOMPATIBLE OBSERVABLES IN QUANTUM MECHANICS

W.M. DE MUYNCK and J.M.V.A. KOELMAN
Department of Theoretical Physics, Eindhoven University of Technology, Eindhoven, The Netherlands

Received 27 July 1983

Starting from a representation of a quantum-mechanical observable by a positive-operator-valued measure it is derived that if two incompatible observables $\mathscr{A}$ and $\mathscr{B}$ are measured jointly then both the $\mathscr{A}$- and the $\mathscr{B}$-measurement are disturbed. More specific: if the expectation values $\langle\mathscr{A}\rangle$ and $\langle\mathscr{B}\rangle$ are undisturbed then there is a broadening in the probability distributions of both the $\mathscr{A}$- and the $\mathscr{B}$-measurement results.

It was demonstrated in ref. [1] on the basis of a quantum-mechanical theory of measurement that the joint measurement of incompatible observables is not excluded by the formalism if it is allowed that the measurements mutually disturb each other. Recently the experimental feasibility of such experiments has been discussed by Yuen [2]. In this note some general properties of joint measurements are derived, starting from the representation [3,4] of a quantum-mechanical observable by a positive-operator-valued measure.

We first consider the measurement of a single observable $\mathscr{A}$. We shall assume that the state of the system can be represented by a density operator ρ on the object Hilbert space $\mathcal{H}$. The observable $\mathscr{A}$ will be represented by the self-adjoint operator A on the same Hilbert space $\mathcal{H}$, with spectral representation

$$A = \sum_m a_m P_m \,. \tag{1}$$

So $\mathscr{A}$ is assumed to possess a discrete but possibly degenerate spectrum. According to ordinary quantum mechanics the probability distribution of the measurement results a_m for the observable $\mathscr{A}$ is given by

$$W_0(a_m;\rho) = \mathrm{Tr}(\rho P_m) \,. \tag{2}$$

We now consider the consequences of a generalisation [3] of the projection operator-valued measures $\{P_m\}$ to positive-operator-valued measures $\{R_m\}$:

$$W(a_m;\rho) = \mathrm{Tr}(\rho R_m) \,, \tag{3}$$

in which R_m are positive self-adjoint operators,

$$R_m \geqslant 0 \,, \tag{4}$$

satisfying

$$\sum_m R_m = I \,, \tag{5}$$

and

$$\sum_m a_m R_m = A \,. \tag{6}$$

By (5) and (6) it is ensured that the probability distribution (3) is normalised, and reproduces the quantum mechanical expectation value $\mathrm{Tr}(\rho A)$ of $\mathscr{A}$. It is not required that this also obtains for all operators $f(\mathscr{A})$, as is the case with probability distribution (2).

The root-mean-square deviation $\Delta_0 \mathscr{A}$ of the measurement results which we would expect to find according to ordinary quantum theory is given by

$$(\Delta_0 \mathscr{A})^2 = \sum_m a_m^2 W_0(a_m;\rho) - \left(\sum_m a_m W_0(a_m;\rho)\right)^2$$

$$= \mathrm{Tr}(\rho A^2) - [\mathrm{Tr}(\rho A)]^2 \,. \tag{7}$$

The root-mean-square deviation $\Delta \mathscr{A}$ of the measurement results represented by (3) can be expressed according to

$$(\Delta \mathscr{A})^2 = \sum_m a_m^2 \, W(a_m; \rho) - \left(\sum_m a_m \, W(a_m; \rho)\right)^2$$

$$= \mathrm{Tr}\left(\rho \sum_m a_m^2 \, R_m\right) - [\mathrm{Tr}(\rho A)]^2 \,. \qquad (8)$$

Combining equations (5), (6), (7) and (8) we obtain

$$(\Delta \mathscr{A})^2 - (\Delta_0 \mathscr{A})^2 = \mathrm{Tr}\left[\rho\left(\sum_m a_m^2 R_m - A^2\right)\right]$$

$$= \sum_m \mathrm{Tr}[\rho(A - a_m) \; R_m(A - a_m)] \geqslant 0 \,, \qquad (9)$$

the non-negativity of the right-hand side being implied by (4) and the consequent positiveness of the operator $(A - a_m)R_m(A - a_m)$. So

$$\Delta \mathscr{A} \geqslant \Delta_0 \mathscr{A} \,. \qquad (10)$$

We now investigate under which conditions the spreading in the measurement results reduces to the minimum spreading. Imposing the requirement

$$\Delta \mathscr{A} = \Delta_0 \mathscr{A} \qquad (11)$$

for all $\rho \geqslant 0$ leads with the help of eq. (9) to the equality

$$(A - a_m)R_m(A - a_m) = 0 \,. \qquad (12)$$

This can easily be shown to imply

$$P_i R_m P_j = 0 \quad \text{for } i \neq m \text{ and } j \neq m \,. \qquad (13)$$

Using requirement (5) we may conclude from (13) that

$$P_m R_m P_m = \sum_j P_m R_j P_m = P_m \,. \qquad (14)$$

But (13) also implies $(I - P_m)R_m(I - P_m) = 0$, whence $R_m(I - P_m) = (I - P_m)R_m = 0$, and

$$P_m R_m P_m = R_m \,. \qquad (15)$$

Comparing this with (14) leads us to the conclusion

$$R_m = P_m \,. \qquad (16)$$

We conclude that the distribution (3) obeying the requirements (4), (5) and (6) meets the requirement of minimum spreading (11) for all density operators ρ if and only if $R_m = P_m$ for all values of m.

With the help of eq. (9) it is easily seen that requirement (11) is equivalent to the relation

$$\sum_m a_m^2 \, R_m = A^2 \,. \qquad (17)$$

Eq. (11) together with the requirements (4), (5) and (6) leads to conclusion (16). Therefore we may state that if the positive operators $\{R_m\}$ satisfy the relation

$$\sum_m a_m^k \, R_m = A^k \quad \text{for} \quad k = 0, 1, \text{and } 2, \qquad (18)$$

then this relation is satisfied for all integers $k \geqslant 0$. We may interpret this result as follows:

If the positive-operator-valued measure $\{R_m\}$ yields the quantum mechanical expectation values of both A and A^2 according to

$$\mathrm{Tr}(\rho A^k) = \sum_m a_m^k \; W(a_m; \rho), \quad k = 1, 2, \qquad (19)$$

then we can calculate the expectation values of all operators A^k according to (19) for $k = 1, 2, 3, \ldots$. A measurement satisfying (19) will be called undisturbed. Such measurements satisfy (16).

It is also observed that (16) can be obtained if, instead of (19), it is required that (4), (5) and (6) hold, and R_m is a projection operator for all m. This result may be inferred from the fact that projection operators R_m, satisfying (5), should necessarily be orthogonal, $R_i R_j = \delta_{ij} R_i$. In view of (6) this leads to the conclusion that $\{R_m\}$ is a spectral representation of the self-adjoint operator A. Since every self-adjoint operator has a unique spectral representation, we arrive at the desired result that (16) must hold for all values of m. It follows that the requirement (18) is equivalent to the requirement that the operators R_m, satisfying (4), (5) and (6), are idempotent.

We now extend the results of the preceding section in a straightforward way to the joint measurement of two observables $\mathscr{A}$ and $\mathscr{B}$ corresponding to self-adjoint operators $A = \Sigma_m a_m P_m$ and $B = \Sigma_n b_n Q_n$, respectively. It is not assumed that A and B commute.

The joint measurement of the observables $\mathcal{A}$ and $\mathcal{B}$ leads to a joint probability distribution of measurement results a_m and b_n,

$$W(a_m, b_n; \rho) = \mathrm{Tr}(\rho R_{mn}) , \tag{20}$$

in which $\{R_{mn}\}$ is a positive-operator-valued measure. The requirements of positivity, normalisation and reproduction of averages may be expressed in the form [cf. eqs. (4)–(6)]:

$$R_{mn} \geqslant 0 , \tag{21}$$

$$\sum_{mn} R_{mn} = I , \tag{22}$$

$$\sum_{mn} a_m R_{mn} = A , \tag{23}$$

$$\sum_{mn} b_n R_{mn} = B . \tag{24}$$

Also the sets $\{\Sigma_n R_{mn}\}$ and $\{\Sigma_m R_{mn}\}$ are positive-operator-valued measures, corresponding to the measurement of observable $\mathcal{A}$ in the presence of the measuring apparatus for observable $\mathcal{B}$, and vice versa. If $\Sigma_n R_{mn} = P_m$ and $\Sigma_m R_{mn} = Q_n$ we shall call the measurements nondisturbing. Then both the $\mathcal{A}$- and $\mathcal{B}$-measurements are undisturbed. In general this is not the case.

We now study the consequences of the requirement that the $\mathcal{A}$-measurement satisfies the minimal spreading condition (11). Repeating the derivation leading up to (16) we arrive at the conclusion that, if

$$\sum_m a_m^k \sum_n R_{mn} = A^k, \quad k = 0, 1, 2, \tag{25}$$

then

$$\sum_n R_{mn} = P_m , \tag{26}$$

and (25) holds for all values of k.

Oncemore, eq. (25) is equivalent to the requirement that the P.O.V. measure $\{\Sigma_n R_{mn}\}$, satisfying (21)–(23), consists of projection operators,

$$\Big(\sum_n R_{mn}\Big)^2 = \sum_n R_{mn} . \tag{27}$$

Lemma. If the self-adjoint operators A and P satisfy the relation $0 \leqslant A \leqslant P = P^2$, then $(I - P)$ is a projection on (a subspace of) the null-space of A. So $(I - P)A = A(I - P) = 0$, which yields $AP = PA = A$.

Considering

$$\Big(\sum_i R_{in}\Big)\Big(\sum_j R_{mj}\Big) = R_{mn}\Big(\sum_j R_{mj}\Big) + \Big(\sum_{i\neq m} R_{in}\Big)I - \Big(\sum_{i\neq m} R_{in}\Big)\Big(\sum_{i\neq m, j} R_{ij}\Big), \tag{28}$$

we may apply the lemma to the right-hand side of (28) because, if (27) obtains, then

$$0 \leqslant R_{mn} \leqslant \sum_j R_{mj} = \Big(\sum_j R_{mj}\Big)^2 ,$$

and

$$0 \leqslant \sum_{i\neq m} R_{in} \leqslant \sum_{i\neq m, j} R_{ij} = \Big(\sum_{i\neq m, j} R_{ij}\Big)^2 .$$

This yields the result

$$\Big(\sum_i R_{in}\Big)\Big(\sum_j R_{mj}\Big) = \Big(\sum_j R_{mj}\Big)\Big(\sum_i R_{in}\Big) = R_{mn} . \tag{29}$$

Using (23) and (24) this gives us

$$AB = BA = \sum_{mn} a_m b_n R_{mn} . \tag{30}$$

Summarising, we may conclude that if, on jointly measuring observables $\mathcal{A}$ and $\mathcal{B}$, it is required that $\mathcal{A}$ has minimal spreading, then the self-adjoint operators A and B should commute. A completely analogous conclusion can be drawn from the requirement of minimal spreading for the observable $\mathcal{B}$. Since minimal spreading implies that the measurement is undisturbed it follows that if $\mathcal{A}$ and $\mathcal{B}$ are incompatible observables *both* the $\mathcal{A}$- and the $\mathcal{B}$-measurement are disturbed if the measurements are performed jointly. This extends a result obtained in ref. [1], where it was demonstrated that at least one of the observables is disturbed. This also partly answers a question raised by Yuen [2] regarding the possibility of the joint measurement of incompatible observables. Evidently, for incompatible observables it is at most possible to measure jointly the expectation values $\langle A \rangle$ and $\langle B \rangle$. It is, however, impossible to derive either $\langle A^2 \rangle$ or $\langle B^2 \rangle$ from the joint probability distribution (20) if A and B do not commute.

The authors wish to express their sincere gratitude to Dr. Roman Groblicki for giving permission to use some of his, as yet, unpublished results.

References

[1] W.M. de Muynck, P.A.E.M. Janssen and A. Santman, Found. Phys. 9 (1979) 71.
[2] H.P. Yuen, Phys. Lett. 91A (1982) 101.
[3] E.B. Davies, Quantum theory of open systems (Academic Press, New York, 1976).
[4] R. Groblicki, Monash University Statistics Research Report No. 53.