

Nonideal Quantum Measurements, Simultaneous Measurement of Incompatible Observables, and the Bell Inequalities

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A partial ordering on the set of quantum mechanical observables is introduced, admitting a natural definition of the concept of (non)ideal measurement. A number of optical detection methods are demonstrated to fit our definition. The set of ideal quantum measurements is determined. The theory is also applied to the simultaneous measurement of incompatible observables. A new inaccuracy relation is presented, limiting the accuracy of such measurements. Examples are given. The existence of a Wigner measure is discussed. The theory is applied to the Bell inequalities. An experiment is discussed in which all four (incompatible) observables are measured simultaneously. The measurement results are predicted to satisfy the Bell inequalities for any state preparation. It is demonstrated how the results of the usual Einstein-Podolsky-Rosen-like experiments can be obtained from the simultaneous measurement ones. The relevance of these results for the significance of the Bell inequalities is discussed.

§1. Introduction

The theory of quantum mechanical measurements has its basis in the axiomatizations by Dirac and von Neumann. In these axiomatizations a quantum mechanical observable is represented by a self-adjoint operator A , having eigenvalues a_k and eigenvectors $|a_k\rangle$ (for simplicity we restrict ourselves to discrete spectra; however, generalization to continuous spectra is often straightforward). If ρ is a density operator describing the state of the object system, then, putting $E_k = |a_k\rangle\langle a_k|$, according to the Dirac-von Neumann axiomatization, the probability of finding the value a_k on measurement of A is $p(a_k) = \text{Tr } \rho E_k$. In this axiomatization the probabilities of observable A are represented by the so called projection valued measure (PVM) $\{E_k\}$.

Mathematically speaking, a PVM is a special case of a more general class of mathematical objects, viz., the positive operator-valued measures (POVM) $\{M_k\}$, $M_k \geq 0$, $\sum_k M_k = I$, in which the operators M_k are not projection operators in general. The preference in axiomatic quantum mechanics for PVMs can be traced back to two possible sources:

- 1) self-adjoint operators as the most direct

generalizations of classical magnitudes;

- 2) interpretation of eigenvalues as properties of the object.

Both issues are combined in the "projection postulate", stating that immediately following a measurement with measurement result a_k the object is in an eigenstate of the projection operator E_k .

We do not want to enter here into the problematic features of the projection postulate. Suffice it to remind that only a restricted class of measurements (so called measurements of the first kind) do satisfy this postulate. Many measurement procedures do not leave the object system in a state prescribed by the projection postulate. Often this final state is completely unknown. Yet, this does not at all obviate such a procedure as a *measurement* procedure, as long as we only require that the procedure yields only information about the object system such as it was at the moment immediately *preceding* the measurement. In general, we need not require that the procedure is *also* preparative (apart from being determinative), since the measured object often gets lost in the measurement process. For this reason we can adopt a more general axiomatization of quantum mechanics, in which

- 1) observables are represented by POVMs;
- 2) measurement results are properties of the measuring instrument (corresponding with pointer positions) rather than properties of the object system.

Of course, the measurement result must be related with some property of the object system. But we make allowance for the possibility that the measurement may yield nonideal or inaccurate information on the object system, because in the amplification process the information may get disturbed. Let us illustrate this by considering the single-mode photon-number counting distribution (during time T)¹⁾

$$p_m(T) = \text{Tr } \rho \mathcal{N} \left\{ \frac{(\xi a^\dagger a)^m}{m!} e^{-\xi a^\dagger a} \right\}, \quad (1)$$

in which ξ is the quantum efficiency of the detector, and \mathcal{N} is the normal ordering operator of the boson annihilation and creation operators a and a^\dagger . For $\xi=1$ this reduces to the expectation value of the PVM $\{E_m = |m\rangle\langle m|\}$, corresponding with the spectral representation of the number operator. This can be interpreted as the result of an *ideal* measurement of the photon number observable. It is not difficult to demonstrate that, for arbitrary ξ in the interval $0 < \xi < 1$, the probability distribution (1) is equal to the expectation value of the POVM

$$\left\{ M_m = \sum_{k=m}^{\infty} \binom{k}{m} \xi^m (1-\xi)^{k-m} E_k \right\}. \quad (2)$$

This demonstrates how the POVM $\{M_m\}$ can be interpreted as describing a nonideal measurement of the photon number observable: if the quantum efficiency ξ is less than 1, the probability $p_m(T) = \text{Tr } \rho M_m$ differs from the ideal value $\text{Tr } \rho E_m$, the difference being attributable to inaccuracy introduced by the measurement procedure. As a matter of fact, the POVM (2) takes into account the fact that some photons that are present, are not registered by the detector.

§2. A Theory of Nonideal Measurements

Generalizing the essential features of the relation (2) between the POVMs $\{M_m\}$ and $\{E_k\}$, we propose²⁾ the following definition of a nonideal measurement:

The POVM $\{M_m\}$ represents a *nonideal*

measurement of the observable (POVM) $\{N_k\}$ (not necessarily a PVM) if the POVMs are related according to

$$M_m = \sum_k \lambda_{mk} N_k, \quad \lambda_{mk} \geq 0, \quad \sum_m \lambda_{mk} = 1. \quad (3)$$

In (3) the stochastic matrix (λ_{mk}) determines the extent to which the measurement $\{M_m\}$ can be considered as an approximate measurement of $\{N_k\}$. It is tempting to interpret λ_{mk} as the conditional probability that, on measurement of $\{M_m\}$, the value m is found if on measurement of $\{N_k\}$ (instead of $\{M_k\}$) the value k would have been found.

It should be noted that our notion of nonideality is a relational notion. It associates two POVMs that both may represent very "nonideal" measurements. Thus, if $N_k = c_k I$, $c_k \geq 0$, $\sum_k c_k = 1$, the probabilities $\text{Tr } \rho N_k$ do not depend on the density operator at all. We call such a POVM an *uninformative* POVM. In this case the POVM $\{M_m\}$ defined by (3) is uninformative too.

We call two POVMs $\{M_m\}$ and $\{N_k\}$ *equivalent* if, apart from (3), we also have

$$N_k = \sum_m \mu_{km} M_m, \quad \mu_{km} \geq 0, \quad \sum_k \mu_{km} = 1. \quad (4)$$

It is clear that (4) can not be satisfied if $\{N_k\}$ is a PVM. It is not difficult, however, to find examples satisfying both (3) and (4) if the N_k operators are not projection operators.

Even if the POVMs $\{M_m\}$ and $\{N_k\}$ are not equivalent, the stochastic matrix (λ_{mk}) may have an inverse (λ_{km}^{-1}) . We then call the measurement $\{M_m\}$ an *invertibly nonideal* measurement of $\{N_k\}$. In this case the probabilities $p_k = \text{Tr } \rho N_k$ can be calculated *exactly* from the measured probabilities $\text{Tr } \rho M_m$ through the relation

$$N_k = \sum_m \lambda_{km}^{-1} M_m. \quad (5)$$

In general (λ_{km}^{-1}) is not a stochastic matrix, the matrix elements not even being nonnegative. It is not devoid of practical importance that the photon detector relation (2) is invertible:

$$E_k = \sum_{m=k}^{\infty} \binom{m}{k} (-1)^{m-k} \xi^{-m} (1-\xi)^{m-k} M_m. \quad (6)$$

Hence, our inefficient photodetector is still able to yield exact information on the photon number.

A measurement represented by a PVM $\{E_k\}$ consisting of *one-dimensional* projection

operators is an ideal measurement in the sense that no inaccuracy is introduced by the measurement procedure as far as the probabilities $\text{Tr } \rho E_k$ are concerned. Mathematically this can be implemented by introducing the notion of a *maximal* POVM:

A POVM $\{M_m\}$ is *maximal* if the existence of a POVM $\{N_k\}$, satisfying (3), implies that POVMs $\{M_m\}$ and $\{N_k\}$ are equivalent. The above mentioned PVMs are not the only POVMs satisfying this definition. We have proved²⁾ that a necessary and sufficient condition for $\{M_m\}$ to be a maximal POVM, is that each operator M_m is *proportional* to a one dimensional projection operator. Apart from the above-mentioned orthogonal resolutions of the identity operator, this also encompasses nonorthogonal resolutions like³⁾

$$M_m = \frac{2}{N} |\psi_m\rangle\langle\psi_m|,$$

$$|\psi_m\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i2\pi(m/N)} \\ e^{i2\pi(m/N)} \end{pmatrix} \in \mathcal{H}_2,$$

$m=1, \dots, N$, N an arbitrary integer, or, for one mode of the electromagnetic field, the POVM

$$M_\alpha = (1/\pi) |\alpha\rangle\langle\alpha|, \quad |\alpha\rangle \text{ a coherent state,}$$

α an arbitrary complex number.

This unification of orthogonal and nonorthogonal resolutions in our opinion is indicative of the general significance of the nonideality structure induced by (3) in the set of positive operator valued measures.

§3. Joint Measurements

Two observables $\{M_m\}$ and $\{N_k\}$ are simultaneously or *jointly measurable* if a bivariate POVM $\{R_{mk}\}$ exists such that

$$\begin{cases} \sum_k R_{mk} = M_m, \\ \sum_m R_{mk} = N_k, \end{cases} \quad (7)$$

and there is a measurement procedure for measuring the observable represented by the POVM $\{R_{mk}\}$.

Two observables $\{M_m\}$ and $\{N_k\}$ are *simultaneously or jointly nonideally measurable* if a bivariate POVM $\{R_{mk}\}$ exists, both in a mathematical and in a physically operational sense, such that the marginals of $\{R_{mk}\}$ are nonideal measurements of $\{M_m\}$ and $\{N_k\}$, i.e., there exist stochastic matrices

(λ_{mn}) and (μ_{kl}) such that

$$\begin{cases} \sum_k R_{mk} = \sum_n \lambda_{mn} M_n, \\ \sum_m R_{mk} = \sum_l \mu_{kl} N_l. \end{cases} \quad (8)$$

We shall be occupied here mainly with the latter situation. Let us first discuss a practical example of a joint nonideal measurement, originally proposed by Busch⁴⁾ but demonstrated here to fit our definition. Consider an incoming one-photon state. Because of the beam splitter *BS* (Fig. 1), with transparency γ , the photon has probability γ to travel in the direction of the polarizer with angle ϑ , and probability $(1-\gamma)$ to go in the other direction, where a polarizer is inserted with a different angle ϑ' . The difference $\vartheta' - \vartheta$ is arbitrary. The polarization observables in the directions ϑ and ϑ' , respectively, are represented by the PVMs (E_+, E_-) and (F_+, F_-) . If $\vartheta' - \vartheta \neq 0$ or $\pi/2$, we have $[E_i, F_j] \neq 0$, $i, j = \pm$. We assume the detector efficiencies of detectors *D* and *D'* to be equal to ξ .

It can be seen as follows that the measurement setup of Fig. 1 can be interpreted as a joint nonideal measurement of the PVMs $\{E_i\}$ and $\{F_j\}$. Denoting by $q(i, j)$ the joint probability of the responses of detectors *D* and *D'*, $(i, j) = (+, +)$ denoting the event that both detectors register a photon, etc., we have the following possibilities:

$$\begin{cases} q(+, +) = 0, \\ q(+, -) = \gamma\xi \text{Tr } \rho E_+, \\ q(-, +) = (1-\gamma)\xi \text{Tr } \rho F_+, \\ q(-, -) = 1 - \gamma\xi \text{Tr } \rho E_+ - (1-\gamma)\xi \text{Tr } \rho F_+. \end{cases} \quad (9)$$

This can be summarized according to $q(i, j) = \text{Tr } \rho R_{ij}$, $i, j = \pm$, the bivariate POVM $\{R_{ij}\}$ being given by

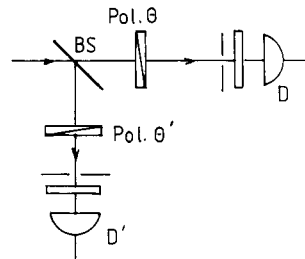


Fig. 1. Joint nonideal measurement of two incompatible polarization observables.

$$\mathbf{R}_{ij} = \begin{pmatrix} 0 & \gamma\xi\mathbf{E}_+ \\ (1-\gamma)\xi\mathbf{F}_+ & \mathbf{I} - \gamma\xi\mathbf{E}_+ - (1-\gamma)\xi\mathbf{F}_+ \end{pmatrix}. \quad (10)$$

It is straightforward to show that the marginals of this POVM satisfy relation (8) according to

$$\begin{aligned} \sum_j \mathbf{R}_{ij} &= \sum_{k=\pm} \lambda_{ik} \mathbf{E}_k, \quad (\lambda_{ik}) = \begin{pmatrix} \gamma\xi & 0 \\ 1-\gamma\xi & 1 \end{pmatrix}, \\ \sum_i \mathbf{R}_{ij} &= \sum_{l=\pm} \mu_{jl} \mathbf{F}_l, \quad (\mu_{jl}) = \begin{pmatrix} (1-\gamma)\xi & 0 \\ 1-(1-\gamma)\xi & 1 \end{pmatrix}. \end{aligned} \quad (11)$$

A second example of a joint nonideal measurement is provided by eight-port homodyning (Fig. 2). The bivariate POVM for this case was essentially derived by Yuen and Shapiro⁵⁾. For one mode of the radiation field it can be demonstrated to fit our definition of a joint nonideal measurement of the incompatible observables $q = (a + a^\dagger)/\sqrt{2}$ and $p = (a - a^\dagger)/i\sqrt{2}$. The measurement inaccuracies of q and p are now described by inaccuracy functions $\lambda(F_1, q)$ and $\mu(F_2, p)$ that turn out to be Gaussians with standard deviations δ_1 and δ_2 , respectively, given by

$$\begin{aligned} \delta_1^2 &= \frac{1}{4(1-\kappa)\xi\varepsilon_1(1-\varepsilon_1)} - 1, \\ \delta_2^2 &= \frac{1}{4\kappa\xi\varepsilon_2(1-\varepsilon_2)} - 1. \end{aligned} \quad (12)$$

Here κ , ε_1 and ε_2 are transparencies of beam splitters in the interferometer (*cf.* Fig. 2), and ξ is, as before, the quantum efficiency of the detectors.

Returning to the general theory, it is interesting to consider the case when both nonideal measurements in (8) are invertible, i.e., the inverses (λ_{mn}^{-1}) and (μ_{kl}^{-1}) of the stochastic matrices exist. In this case it is possible to *calculate* both probability distributions $\text{Tr } \rho \mathbf{M}_m$ and $\text{Tr } \rho \mathbf{N}_k$ from the measured probability distribution $\text{Tr } \rho \mathbf{R}_{mk}$. This simply follows from the inversion of the relations in (8). A somewhat more sophisticated way of dealing with this invertibility is, to define a so called Wigner measure $\{\mathbf{W}_{mk}\}$ according to

$$\mathbf{W}_{mk} = \sum_{nl} \lambda_{mn}^{-1} \mu_{kl}^{-1} \mathbf{R}_{nl}, \quad (13)$$

which has the POVMs $\{\mathbf{M}_m\}$ and $\{\mathbf{N}_k\}$ as its marginals.

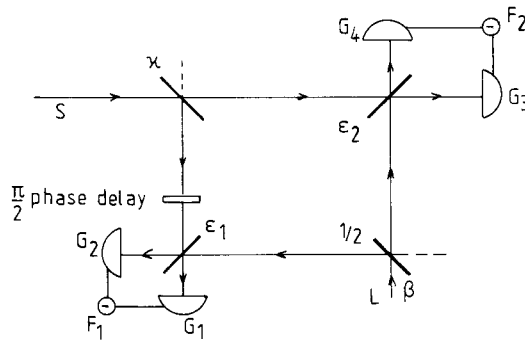


Fig. 2. Eight-port optical homodyning as a joint measurement of q and p .

It turns out that in case of invertibility the joint nonideal measurement setup yields the probability distributions of *both* incompatible observables exactly, even though the measurement procedure may disturb the correlation between the two. Although on the level of individual measurement results the Bohr-Heisenberg thesis of the impossibility of neglecting or compensating for the interaction between object and measuring instrument remains true, it turns out that, under certain conditions, such a compensation is possible on the level of the statistical data.

For the joint polarization measurement, discussed above, we obtain from (11):

$$\begin{aligned} (\lambda_{ki}^{-1}) &= \begin{pmatrix} (\gamma\xi)^{-1} & 0 \\ 1 - (\gamma\xi)^{-1} & 1 \end{pmatrix}, \\ (\mu_{lj}^{-1}) &= \begin{pmatrix} (1-\gamma)^{-1}\xi^{-1} & 0 \\ 1 - (1-\gamma)^{-1}\xi^{-1} & 1 \end{pmatrix} \end{aligned}$$

and

$$(\mathbf{W}_{mk}) = \begin{pmatrix} 0 & \mathbf{E}_+ \\ \mathbf{F}_+ & \mathbf{I} - \mathbf{E}_+ - \mathbf{F}_+ \end{pmatrix}.$$

It can be shown that, for one mode of the radiation field, eight-port homodyning is completely analogous, the expectation value of the Wigner measure being equal to the well-known Wigner distribution.

§4. Limitations on Joint Measurement

The Heisenberg indeterminacy relations are often interpreted as restricting the possibility of the simultaneous measurement of incompatible observables because of mutual disturbance of the measurement procedures. This in-

terpretation, however, is not uncontroversial. Since the Heisenberg relation can be tested by means of separate measurements of each of the two (possibly maximal) observables, an interpretation as a restriction on the *preparation* of the object system⁶⁾ seems to be more plausible. Yet, the thought experiments advanced by Bohr and Heisenberg to support their interpretation indicate that the idea of a mutual disturbance is not illusory at all.

Our formalism seems to comply with this idea. Thus, for the joint polarization measurement (11), comparing the nonideal measurements represented by the stochastic matrices (λ_{ik}) and (μ_{jl}) with an ideal measurement (corresponding with a stochastic matrix that is equal to the unit matrix), we can define as a nonideality measure for each of the two nonideal measurements the difference δ between the products of the eigenvalues of the unit matrix and those of the nonideality matrix, yielding $\delta_{(i)} = 1 - \gamma\xi$, $\delta_{(j)} = 1 - (1 - \gamma)\xi$, and implying the inaccuracy relation

$$\delta_{(i)} + \delta_{(j)} = 2 - \xi \geq 1.$$

For eight-port homodyning we obtain from (12) the inaccuracy relation

$$\delta_1 \cdot \delta_2 \geq \frac{2}{\xi} - 1 \geq 1.$$

In the following a new inequality is introduced that is generally applicable to the joint measurement of two maximal PVMs $\{E_k\}$ and $\{F_l\}$ on a finite (n -) dimensional Hilbert space. Given that the POVM $\{M_m\}$ is a nonideal measurement of $\{E_k\}$, we define the following nonideality measure for the relation between the two POVMs, based on the notion of mutual information,⁷⁾ according to:

$$I_{\{E_k\} \rightarrow \{M_m\}} = \sum_{m,k} \lambda_{mk} p_k \log \frac{\lambda_{mk}}{\sum_{k'} \lambda_{mk'} p_{k'}},$$

$$p_k = n^{-1} \text{Tr } M_k. \tag{14}$$

Note that the quantity defined by (14) does not depend on the density operator, and, hence, is independent of the preparation. It is solely a property of the relation between the two measurement procedures corresponding with the POVMs $\{M_m\}$ and $\{E_k\}$. For an ideal measurement we get the maximal value $\log n$ for $I_{\{E_k\} \rightarrow \{M_m\}}$. So we may define

$$\delta_{\{E_k\} \rightarrow \{M_m\}} = \log n - I_{\{E_k\} \rightarrow \{M_m\}}$$

as a nonideality measure. Now, assuming $\text{Tr } E_k F_l \neq 0$ for all k, l , it is possible⁸⁾ to derive the following inequality if $\{R_{ij}\}$ is a joint nonideal measurement of both $\{E_k\}$ and $\{F_l\}$:

$$\delta_{\{E_k\} \rightarrow \{\sum_j R_{ij}\}} + \delta_{\{F_l\} \rightarrow \{\sum_j R_{ij}\}} \geq -\log (\max_{k,l} \text{Tr } E_k F_l). \tag{15}$$

This relation is independent of ρ , and, hence, independent of the preparation. It sets a lower limit to the measurement accuracies that can be obtained in a joint nonideal measurement of incompatible observables. Hence, relation (15) is a genuine inaccuracy inequality, to be distinguished conceptually from the Heisenberg uncertainty relation. It was demonstrated⁸⁾ that a POVM $\{R_{ij}\}$ exists for every pair of PVMs $\{E_k\}$ and $\{F_l\}$ on a finite dimensional Hilbert space.

§5. Bell Inequalities Without Hidden Variables

Usually the Bell inequalities (BI) are derived within the context of (local) hidden variables theories in order to demonstrate that quantum mechanics is at variance with such theories. It, however, is argued^{9,10)} that not (local) hidden variables are the essential ingredients of this derivation, but the existence of a quadrivariate joint probability distribution (qjpd) of the observables that are involved. Our theory of joint (nonideal) measurement of incompatible observables contains such qjpd's.

Consider a correlated photon pair as used, e.g., in the experiments of Aspect *et al.*¹¹⁾ On one photon the measurement of Fig. 1 is performed (transparency γ_1 , angles ϑ_1 and ϑ'_1 , detectors D_1 and D'_1). On the other photon we can perform a similar experiment (transparency γ_2 , angles ϑ_2 and ϑ'_2 , detectors D_2 and D'_2). Analogously to (9) it is easy to see that for this experiment a qjpd, describing the joint detection probability of detectors D_1, D'_1, D_2 and D'_2 is given by

$$q(i, j, k, l) = \text{Tr } \rho R_{ij}^1 R_{kl}^2, \tag{16}$$

in which $R_{ij}^w, w=1,2$, are given by (10). The existence of the qjpd (16) implies that the BI are satisfied for *any* density operator ρ .

It is interesting to note that for each arm of

the interferometer, we can calculate a Wigner measure $\{W_{mk}^w\}$, $w=1,2$, analogous to (13). This yields the possibility to calculate from the qjpd $q(i, j, k, l)$ the quasi-probability distribution $\text{Tr } \rho W_{ij}^1 W_{kl}^2$, having as marginals the bivariate jpd's $\text{Tr } \rho E_i^1 E_k^2$, $\text{Tr } \rho E_i^1 F_l^2$, etc., that are obtained in the usual EPR like experiments measuring jointly two compatible observables at a time, like the ones performed by Aspect *et al.* Hence, in a way in our experiment the four usual EPR-like experiments are performed at the same time.

We like to finish this contribution by pointing at the possible relevance of our experiment for the interpretation of quantum mechanics. It is demonstrated above that the difference between experiments satisfying and those violating the BI can be described by the transition from the POVM $M_{ij}^1 M_{kl}^2$ to the Wigner measure $W_{ij}^1 W_{kl}^2$. It should be emphasized that this transition is carried out by a change in each of the arms of the interferometer separately. There is no nonlocality involved in this transition, as also is embodied in the fact that both the M_{ij}^w and the W_{ij}^w operators satisfy the principle of local commutativity.

As we saw earlier the transition between

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O. Hirota: Homodyne detection is not example of operator-valued measure (OVM). It corresponds to a projection valued measure. Rather, Heterodyne detection is an example of OVM, because it measures non-commutative observables: complex amplitude of light wave.

W. M. de Muynck: I completely agree with this question

$\{M_{ij}^w\}$ and $\{W_{ij}^w\}$, for a definite value of w , takes into account the mutual disturbance in jointly measuring two incompatible observables. For this reason it seems to us that the violation of the BI in quantum mechanics should be attributed to the incompatibility of observables of the *same* photon rather than to some nonlocal influence in measuring two *compatible* observables of *different* photons.

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as far as four-port homodyning is concerned. This, indeed, corresponds with a projection-valued measure. The example I consider, however, is eight-port homodyning which was demonstrated by Yuen and Shapiro to correspond with a positive operator-valued measure.