Joint measurement of interference and path observables in optics and neutron interferometry

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We discuss analogous measurement arrangements for the joint measurement of interference and path in both optical and neutron interferometry. Analogous to an optical experiment proposed by Busch we propose a neutron interferometric experiment that, contrary to the ones performed till now, is a nontrivial joint measurement of interference and path.

In the Dirac-von Neumann formulation of quantum mechanics a quantum mechanical observable is described by a self-adjoint operator or rather by its orthogonal spectral resolution which defines a projection-valued measure (PVM). The expectation values of the operators of this PVM represent the probabilities of the measurement results. In this formalism it is impossible to describe the joint measurement of incompatible observables like interference and path. It is possible to do so in the generalized formalism of positive operator-valued measures (POVM) first developed by Davies [1]. On this basis a theory of joint measurement of incompatible observables was developed by Martens and de Muynck [2].

Let \{Q_k\} and \{P_n\} be two incompatible POVMS, i.e. \([Q_k, P_n]\neq 0\) in general. Then we shall say that \{Q_k\} and \{P_n\} are jointly nonideally measurable if a measurement procedure exists, the measurement probabilities of which are described by the expectation values of a bivariate POVM \{M_{ij}\}, related to the POVMS \{Q_k\} and \{P_n\} in the following way:

\[
\sum_j M_{ij} = \sum_n \lambda_{in} P_n, \quad \lambda_{in} \geq 0, \quad \sum_i \lambda_{in} = 1,
\]

\[
\sum_i M_{ij} = \sum_k \lambda_{jk} Q_k, \quad \mu_{jk} \geq 0, \quad \sum_j \mu_{jk} = 1.
\]

If the conditions (1) are satisfied, the measurement procedure for the observable \(M_{ij}\) yields, through its marginals \(\sum_j M_{ij}\) and \(\sum_i M_{ij}\), a certain amount of information on the observables \{Q_k\} and \{P_n\}, this amount becoming larger as the matrices \((\lambda_{in})\) and \((\mu_{jk})\) approach the unit matrix. For this reason these matrices are called nonideality matrices. These matrices were demonstrated [2] to satisfy a relation of complementarity such that, if \{Q_k\} and \{P_n\} are incompatible, then one matrix must become more nonideal as the other one approaches the unit matrix.

It is possible that the nonideality matrices have inverses. Then (1) can be inverted, yielding

\[
P_n = \sum_i (\lambda^{-1})_{ni} \sum_j M_{ij},
\]

\[
Q_k = \sum_j (\mu^{-1})_{kj} \sum_i M_{ij}.
\]

If such inversion is possible the quantum mechanical probabilities of both incompatible observables \{Q_k\} and \{P_n\} can be calculated from the probabilities obtained in the measurement of observable \{M_{ij}\}. In this case it is possible to define the Wigner measure

\[
W_{nk} := \sum_{ij} (\lambda^{-1})_{ni} (\mu^{-1})_{kj} M_{ij},
\]
satisfying
\[ \sum_k W_{nk} = P_n, \]
\[ \sum_n W_{nk} = Q_k. \] (4)

Note that the operators \( W_{nk} \) of the Wigner measure need not be positive.

In this paper we shall discuss a number of optical and neutron interference experiments satisfying (1) and (2) for \( \{Q_k\} \) an interference observable and \( \{P_n\} \) the observable corresponding with a determination of the path the photon or neutron took through the interferometer. It will be demonstrated that the analogy between optical and neutron interference is persistent under the generalization of the formalism.

Consider the Mach-Zehnder interferometer of fig. 1. Beam splitters \( B_1 \) and \( B_2 \) have transparancies \( 1/2 \). An extra beam splitter \( B_3 \) is inserted, having transparency \( \alpha \). In one arm the photon undergoes a phase shift \( \chi \). If \( \alpha = 1 \), and detectors \( D_i \) have efficiencies \( \xi = 1 \), the detection probabilities of \( D_1 \) and \( D_2 \) are given by \( p_k = \langle Q_k \rangle, \ k = 1, 2, \)
\[ Q_1 = \begin{pmatrix} \cos^2 \chi & -i \sin \chi \sin \chi \\ i \frac{1}{2} \sin \chi & \sin^2 \chi \end{pmatrix}, \ Q_2 = I - Q_1, \] (5)
the expectation values being taken in the incoming state, which may be a superposition of one-photon states of the incoming modes 1 and 2 (cf. fig. 1) by which the two-dimensional representation is defined (in most actual experiments only one of these modes is fed at a time). We call the POVM \( \{Q_1, Q_2\} \) the interference observable. It is a projection-valued measure. If \( \alpha = 0 \) the path of the photon is measured, the detection probabilities \( p_1 + p_2 \) and \( p_3 \) being given as \( \langle P_n \rangle, \ n = \text{"left" or "right"}, \) respectively, and
\[ P_l = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ P_r = I - P_l. \] (6)

Figure 1: Trivial joint measurement of optical interference and path. Optical Mach-Zehnder interferometer; \( D_i \) = optical detectors, \( B_i \) = beam splitters; \( \alpha \) = transparancy of \( B_3 \); \( \chi \) = phase shift.
The PVM \( \{ P_l, P_r \} \) is called the path observable. For \( 0 < \alpha < 1 \) we can calculate\(^3\) the detection probabilities \( p_i = \langle M_i \rangle \) of \( D_i, \ i = 1, 2, 3 \), obtaining
\[
M_1 = \frac{1}{2} [ P_l + \alpha P_r + \sqrt{\alpha}(Q_1 - Q_2) ],
\]
\[
M_2 = \frac{1}{2} [ P_l + \alpha P_r - \sqrt{\alpha}(Q_1 - Q_2) ],
\]
\[
M_3 = (1 - \alpha) P_r .
\]

In order to demonstrate that the POVM \( \{ M_1, M_2, M_3 \} \) can be interpreted as a joint measurement of interference and path we order it into a bivariate POVM according to
\[
(M_{ij}) = \begin{pmatrix} M_1, & M_2, \\ \epsilon M_3, & (1 - \epsilon) M_3 \end{pmatrix} .
\]

If we choose \( \epsilon = 1/2 \) the marginals of this bivariate POVM can be calculated as
\[
\begin{pmatrix} M_1 + M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} \lambda_{ll} & \lambda_{lr} \\ \lambda_{rl} & \lambda_{rr} \end{pmatrix} \begin{pmatrix} P_l \\ P_r \end{pmatrix},
\]
\[
\begin{pmatrix} M_1 + \epsilon M_3 \\ M_2 + (1 - \epsilon) M_3 \end{pmatrix} = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},
\]
the matrices \( (\lambda_{in}) \) and \( (\mu_{jk}) \) being given by
\[
(\lambda_{in}) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 - \alpha \end{pmatrix},
\]
\[
(\mu_{jk}) = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{\alpha} & 1 - \sqrt{\alpha} \\ 1 - \sqrt{\alpha} & 1 + \sqrt{\alpha} \end{pmatrix}.
\]

Hence, the experiment satisfies definition (1) for the joint nonideal measurement of interference and path. It is also easy to see that both nonideality matrices \( (\lambda_{in}) \) and \( (\mu_{jk}) \) can be inverted. Hence, also (2) is satisfied. The Wigner measure (3) for this experiment is found to be
\[
W_{11} = \frac{1}{2} (P_l + Q_1 - Q_2),
\]
\[
W_{12} = \frac{1}{2} (P_l - Q_1 + Q_2),
\]
\[
W_{21} = W_{22} = \frac{1}{2} P_r ,
\]
which is directly seen to satisfy (4).

The measurement arrangement of fig. 1 can be changed so as to allow the semitransparent mirror \( B_2 \) to have arbitrary transparency \( \gamma \). The measurement can, also in this general case, be interpreted as a joint nonideal measurement of interference and path, i.e. eqs. (9) and (10) remain valid, if \( \epsilon \) is chosen\(^3\) according to
\[
\epsilon = \frac{1 - \sqrt{\alpha} - \gamma(1 - \sqrt{\alpha})^2}{1 - \alpha}.
\]
The nonideality matrix \((\mu_{jk})\), in the general case, becomes

\[
(\mu_{jk}) = \begin{pmatrix}
1 - \gamma(1 - \sqrt{\alpha}) & (1 - \gamma)(1 - \sqrt{\alpha}) \\
\gamma(1 - \sqrt{\alpha}) & 1 - (1 - \gamma)(1 - \sqrt{\alpha})
\end{pmatrix},
\]

(15)

the nonideality matrix \((\lambda_{in})\) remaining unchanged equal to (11).

In fig. 2 the measurement set-up for neutrons is given which performs the analogous experiment of fig. 1. The extra beam splitter \(B_3\) finds its analogue in an absorber (with transmission probability \(\alpha\)) placed in one of the beams, and intended to yield information on the path the neutron has taken [4]. If a stochastic absorber is taken, then essentially the same POVM \(\{M_1, M_2, M_3\}\) (7) is obtained [5], \(\langle M_3 \rangle\) being the probability that the neutron is absorbed in the absorber. All results as regards the interpretation of the experiment as a joint measurement of interference and path can be taken over from the optical case.

An alternative experiment exploits deterministic rather than stochastic absorption. In this experiment the stochastic absorber is replaced by a beam chopper, that either leaves the particle completely undisturbed (probability \(v\)) or absorbs it with certainty. This experiment can be described by a POVM obtained from (7) in the following manner:

\[
M'_i = vM_i(\alpha = 1) + (1 - v)M_i(\alpha = 0), \quad i = 1, 2, 3.
\]

(16)

This POVM essentially describes the results of an experiment in which the probability distribution \(\langle M'_i \rangle\) is a weighted mean of results obtained in a pure interference measurement (weight \(v\)) and a pure path measurement (weight \((1 - v)\)). It is obvious that also this experiment can be done easily in the optical case by replacing the semitransparent mirror \(B_3\) in fig. 1 by a rotating beam chopper that is perfectly reflecting if the photon beam is blocked by it.

All experiments discussed till now are so-called trivial joint measurements because they do not yield excess information on the state of the incoming object over the one obtained by combining the results of the two pure measurements. This is the case if and only if the elements \(M_{ij}\) of the POVM \(\{M_{ij}\}\) can be written as linear combinations of the elements of the PVMs \(\{Q_1, Q_2\}\) and \(\{P_l, P_r\}\):

\[
M_{ij} = \sum_k c_{kij}Q_k + \sum_n d_{nij}P_n,
\]

(17)
because in this case the joint probability distribution $\langle M_{ij} \rangle$ can be calculated from the measurement results of the pure path and interference measurements.

By Busch [6] an optical experiment is proposed that is nontrivial in this respect (cf. fig. 3). Taking transparencies and phase shifts as indicated in the figure the detection probabilities of detectors $D_i$, $i = 1, \ldots, 4$, are represented by the following POVM $\{R_i\}$:

\[
R_1 = \frac{1}{2}[(1-\alpha)P_l + \alpha P_r + \sqrt{\alpha(1-\alpha)}(Q_1 - Q_2)],
R_2 = \frac{1}{2}[(1-\alpha)P_l + \alpha P_r - \sqrt{\alpha(1-\alpha)}(Q_1 - Q_2)],
R_3 = \frac{1}{2}[\alpha P_l + (1-\alpha)P_r + \sqrt{\alpha(1-\alpha)}(Q_1' - Q_2')],
R_4 = \frac{1}{2}[\alpha P_l + (1-\alpha)P_r - \sqrt{\alpha(1-\alpha)}(Q_1' - Q_2')].
\]

(18)

Here $Q_1'$ and $Q_2'$ are defined by substituting in (5) $\chi'$ for $\chi$. Arranging this POVM as a bivariate POVM according to

\[
(M_{ij}) = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix},
\]

(19)

we obtain for the marginals

\[
\begin{pmatrix} R_1 + R_2 \\ R_3 + R_4 \end{pmatrix} = \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} P_l \\ P_r \end{pmatrix},
\]

\[
\begin{pmatrix} R_1 + R_2 \\ R_3 + R_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \sqrt{\alpha(1-\alpha)} \cos \frac{\chi+\chi'}{2} \\ \frac{1}{2} - \sqrt{\alpha(1-\alpha)} \cos \frac{\chi-\chi'}{2} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \begin{pmatrix} \frac{\chi+\chi'}{2} \\ \frac{\chi-\chi'}{2} \end{pmatrix}
\]

(20)

indicating that the measurement can be interpreted as a joint measurement of path and interference with phase shift $(\chi + \chi')/2$. 

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**Figure 3:** Nontrivial joint measurement of optical interference and path. Transparencies of beam splitters as indicated; $\chi, \chi'$ = phase shifts.
An analogous neutron interference experiment is schematically given in fig. 4. We need a four-slab interferometer instead of the three-slab one that is usually employed. The elements indicated by α and 1 - α, respectively, are scatterers for which the differential cross-sections in the indicated directions are such that they act like the optical beam splitters of transparencies α and 1 - α. Calculation of the POVM describing this experiment demonstrates the complete analogy with the optical experiment of fig. 3, the POVM being given essentially by (18).

Inverting the nonideality matrices exhibited in (20) (which is possible if α ≠ 1/2 and |χ - χ'| ≠ π), we obtain from (3) the Wigner measure for these experiments:

\[
\begin{align*}
W_{11} &= \frac{1}{2} P_l + \frac{(Q_1 - Q_2)}{4(1-2\alpha)\cos \frac{x}{2-x'}} - \frac{\alpha(Q_1(x+y')/2 - Q_2(x+y'))}{2(1-2\alpha)}, \\
W_{12} &= \frac{1}{2} P_l - \frac{(Q_1 - Q_2)}{4(1-2\alpha)\cos \frac{x}{2-x'}} + \frac{\alpha(Q_1(x+y')/2 - Q_2(x+y'))}{2(1-2\alpha)}, \\
W_{21} &= \frac{1}{2} P_r + \frac{(Q_1 - Q_2)}{4(1-2\alpha)\cos \frac{x}{2-x'}} - \frac{\alpha(Q_1(x+y')/2 - Q_2(x+y'))}{2(1-2\alpha)}, \\
W_{22} &= \frac{1}{2} P_r - \frac{(Q_1 - Q_2)}{4(1-2\alpha)\cos \frac{x}{2-x'}} + \frac{\alpha(Q_1(x+y')/2 - Q_2(x+y'))}{2(1-2\alpha)}. \\
\end{align*}
\]

The marginals of this Wigner measure can easily be found as the PVMs \{P_l, P_r\} and \{Q_1(χ + χ')/2, Q_2(χ + χ')/2\}. The nontriviality of this measurement is reflected in the fact that (21) is not of the general form (17) (as is (13)). For this reason the experiments of figs. 3 and 4 have a larger capacity of discriminating between different incoming states than the experiments of figs. 1 and 2.

References