Joint measurement of interference and path observables in optics and neutron interferometry[1]

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We discuss analogous measurement arrangements for the joint measurement of interference and path in both optical and neutron interferometry. Analogous to an optical experiment proposed by Busch we propose a neutron interferometric experiment that, contrary to the ones performed till now, is a nontrivial joint measurement of interference and path.

In the Dirac-von Neumann formulation of quantum mechanics a quantum mechanical observable is described by a self-adjoint operator or rather by its orthogonal spectral resolution which defines a projection-valued measure (PVM). The expectation values of the operators of this PVM represent the probabilities of the measurement results. In this formalism it is impossible to describe the joint measurement of incompatible observables like interference and path. It is possible to do so in the generalized formalism of positive operator-valued measures (POVM) first developed by Davies [1]. On this basis a theory of joint measurement of incompatible observables was developed by Martens and de Muynck [2].

Let $\{\mathbf{Q}_k\}$ and $\{\mathbf{P}_n\}$ be two incompatible POVMS, i.e. $[\mathbf{Q}_k, \mathbf{P}_n] \neq 0$ in general. Then we shall say that $\{\mathbf{Q}_k\}$ and $\{\mathbf{P}_n\}$ are jointly nonideally measurable if a measurement procedure exists, the measurement probabilities of which are described by the expectation values of a bivariate POVM $\{\mathbf{M}_{ij}\}$, related to the POVMS $\{\mathbf{Q}_k\}$ and $\{\mathbf{P}_n\}$ in the following way:

$$\begin{array}{l} \sum_j \mathbf{M}_{ij} = \sum_n \lambda_{in}\mathbf{P}_n,\ \lambda_{in} \geq 0,\ \sum_i \lambda_{in} = 1, \\ \\ \sum_i \mathbf{M}_{ij} = \sum_k \lambda_{jk}\mathbf{Q}_k,\ \mu_{jk} \geq 0,\ \sum_j \mu_{jk} = 1. \end{array} \tag{1}$$

If the conditions (1) are satisfied, the measurement procedure for the observabie $\mathbf{M}_{ij}$ yields, through its marginals $\sum_j \mathbf{M}_{ij}$ and $\sum_i \mathbf{M}_{ij}$, a certain amount of information on the observabies $\{\mathbf{Q}_k\}$ and $\{\mathbf{P}_n\}$, this amount becoming larger as the matrices (λ_{in}) and (μ_{jk}) approach the unit matrix. For this reason these matrices are called nonideality matrices. These matrices were demonstrated [2] to satisfy a relation of complementarity such that, if $\{\mathbf{Q}_k\}$ and $\{\mathbf{P}_n\}$ are incompatible, then one matrix must become more nonideal as the other one approaches the unit matrix.

It is possible that the nonideality matrices have inverses. Then (1) can be inverted, yielding

$$\begin{array}{l} \mathbf{P}_n = \sum_i (\lambda^{-1})_{ni} \sum_j \mathbf{M}_{ij}, \\ \\ \mathbf{Q}_k = \sum_j (\mu^{-1})_{kj} \sum_i \mathbf{M}_{ij}. \end{array} \tag{2}$$

If such inversion is possible the quantum mechanical probabilities of both incompatible observables $\{\mathbf{Q}_k\}$ and $\{\mathbf{P}_n\}$ can be calculated from the probabilities obtained in the measurement of observable $\{\mathbf{M}_{ij}\}$. In this case it is possible to define the Wigner measure

$$\mathbf{W}_{nk} := \sum_{ij} (\lambda^{-1})_{ni} (\mu^{-1})_{kj} \mathbf{M}_{ij}, \tag{3}$$

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satisfying

$$\begin{array}{l}\sum_k \mathbf{W}_{nk} = \mathbf{P}_n,\\ \sum_n \mathbf{W}_{nk} = \mathbf{Q}_k.\end{array} \tag{4}$$

Note that the operators $\mathbf{W}_{nk}$ of the Wigner measure need not be positive.

In this paper we shall discuss a number of optical and neutron interference experiments satisfying (1) and (2) for $\{\mathbf{Q}_k\}$ an interference observable and $\{\mathbf{P}_n\}$ the observable corresponding with a determination of the path the photon or neutron took through the interferometer. It will be demonstrated that the analogy between optical and neutron interference is persistent under the generalization of the formalism.

Consider the Mach-Zehnder interferometer of fig. 1. Beam splitters B_1 and B_2 have transparancies 1/2. An extra beam splitter B_3 is inserted, having transparancy α. In one arm the photon undergoes a phase shift χ. If $\alpha = 1$, and detectors D_i have efficiencies $\xi = 1$, the detection probabilities of D_1 and D_2 are given by $p_k = \langle \mathbf{Q}_k \rangle,\ k = 1, 2,$

$$\mathbf{Q}_1 = \begin{pmatrix} \cos^2 \frac{\chi}{2} & -\frac{i}{2}\sin\chi \\ \frac{i}{2}\sin\chi & \sin^2\frac{\chi}{2} \end{pmatrix},\ \mathbf{Q}_2 = \mathbf{I} - \mathbf{Q}_1, \tag{5}$$

the expectation values being taken in the incoming state, which may be a superposition of one-photon states of the incoming modes 1 and 2 (cf. fig. 1) by which the two-dimensional representation is defined (in most actual experiments only one of these modes is fed at a time). We call the POVM $\{\mathbf{Q}_1, \mathbf{Q}_2\}$ the interference observabie. It is a projection-valued measure. If $\alpha = 0$ the path of the photon is measured, the detection probabilities $p_1 + p_2$ and p_3 being given as $\langle \mathbf{P}_n \rangle,\ n =$ "left" or "right", respectively, and

$$\mathbf{P}_l = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ \mathbf{P}_r = \mathbf{I} - \mathbf{P}_l. \tag{6}$$

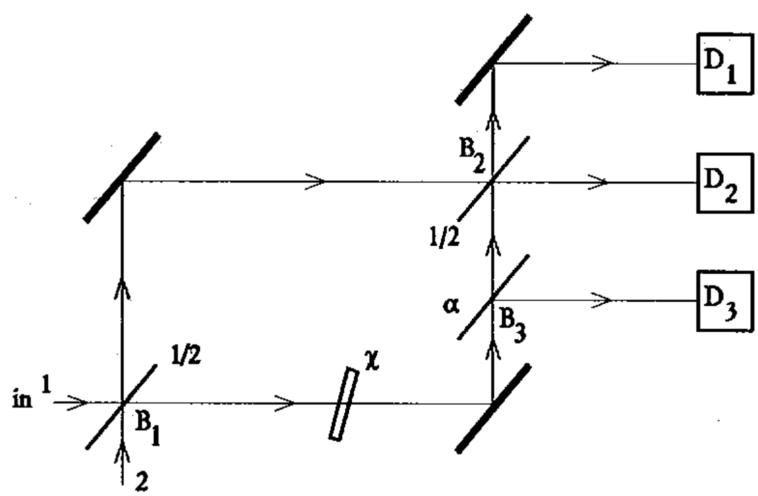

Figure 1: Trivial joint measurement of optical interference and path. Optical Mach-Zehnder interferometer; D_i = optical detectors, B_i=beam splitters; α = transparancy of B_3; χ = phase shift.

The PVM $\{\mathbf{P}_l, \mathbf{P}_r\}$ is called the path observable. For $0 < \alpha < 1$ we can calculate [3] the detection probabilities $p_i = \langle \mathbf{M}_i \rangle$ of $\mathrm{D}_i,\ i = 1, 2, 3$, obtaining

$$\begin{aligned}
\mathbf{M}_1 &= \tfrac{1}{2}[\mathbf{P}_l + \alpha \mathbf{P}_r + \sqrt{\alpha}(\mathbf{Q}_1 - \mathbf{Q}_2)], \\
\mathbf{M}_2 &= \tfrac{1}{2}[\mathbf{P}_l + \alpha \mathbf{P}_r - \sqrt{\alpha}(\mathbf{Q}_1 - \mathbf{Q}_2)], \\
\mathbf{M}_3 &= (1-\alpha)\mathbf{P}_r.
\end{aligned} \tag{7}$$

In order to demonstrate that the POVM $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ can be interpreted as a joint measurement of interference and path we order it into a bivariate POVM according to

$$(\mathbf{M}_{ij}) = \begin{pmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \epsilon \mathbf{M}_3 & (1-\epsilon)\mathbf{M}_3 \end{pmatrix}. \tag{8}$$

If we choose $\epsilon = 1/2$ the marginals of this bivariate POVM can be calculated as

$$\begin{pmatrix} \mathbf{M}_1 + \mathbf{M}_2 \\ \mathbf{M}_3 \end{pmatrix} = \begin{pmatrix} \lambda_{ll} & \lambda_{lr} \\ \lambda_{rl} & \lambda_{rr} \end{pmatrix} \begin{pmatrix} \mathbf{P}_l \\ \mathbf{P}_r \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} \mathbf{M}_1 + \epsilon \mathbf{M}_3 \\ \mathbf{M}_2 + (1-\epsilon)\mathbf{M}_3 \end{pmatrix} = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{pmatrix}, \tag{10}$$

the matrices (λ_{in}) and (μ_{jk}) being given by

$$(\lambda_{in}) = \begin{pmatrix} 1 & \alpha \\ 0 & 1-\alpha \end{pmatrix}, \tag{11}$$

$$(\mu_{jk}) = \frac{1}{2} \begin{pmatrix} 1+\sqrt{\alpha} & 1-\sqrt{\alpha} \\ 1-\sqrt{\alpha} & 1+\sqrt{\alpha} \end{pmatrix}. \tag{12}$$

Hence, the experiment satisfies definition (1) for the joint nonideal measurement of interference and path. It is also easy to see that both nonideality matrices (λ_{in}) and (μ_{jk}) can be inverted. Hence, also (2) is satisfied. The Wigner measure (3) for this experiment is found to be

$$\begin{aligned}
\mathbf{W}_{11} &= \tfrac{1}{2}(\mathbf{P}_l + \mathbf{Q}_1 - \mathbf{Q}_2), \\
\mathbf{W}_{12} &= \tfrac{1}{2}(\mathbf{P}_l - \mathbf{Q}_1 + \mathbf{Q}_2), \\
\mathbf{W}_{21} &= \mathbf{W}_{22} = \tfrac{1}{2}\mathbf{P}_r,
\end{aligned} \tag{13}$$

which is directly seen to satisfy (4).

The measurement arrangement of fig. 1 can be changed so as to allow the semitransparant mirror B_2 to have arbitrary transparancy γ. The measurement can, also in this general case, be interpreted as a joint nonideal measurement of interference and path, i.e. eqs. (9) and (10) remain valid, if ϵ is chosen [3] according to

$$\epsilon = \frac{1-\sqrt{\alpha}-\gamma(1-\sqrt{\alpha})^2}{1-\alpha}. \tag{14}$$

The nonideality matrix (μ_{jk}), in the general case, becomes

$$(\mu_{jk}) = \begin{pmatrix} 1-\gamma(1-\sqrt{\alpha}) & (1-\gamma)(1-\sqrt{\alpha}) \\ \gamma(1-\sqrt{\alpha}) & 1-(1-\gamma)(1-\sqrt{\alpha}) \end{pmatrix}, \tag{15}$$

the nonideality matrix (λ_{in}) remaining unchanged equal to (11).

In fig. 2 the measurement set-up for neutrons is given which performs the analogous experiment of fig. 1. The extra beam splitter B_3 finds its analogue in an absorber (with transmission probability α) placed in one of the beams, and intended to yield information on the path the neutron has taken [4]. If a stochastic absorber is taken, then essentially the same POVM $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ (7) is obtained [5], $\langle \mathbf{M}_3 \rangle$ being the probability that the neutron is absorbed in the absorber. All results as regards the interpretation of the experiment as a joint measurement of interference and path can be taken over from the optical case.

An alternative experiment exploits *deterministic* rather than stochastic absorption. In this experiment the stochastic absorber is replaced by a beam chopper, that either leaves the particle completely undisturbed (probability v) or absorbs it with certainty. This experiment can be described by a POVM obtained from (7) in the following manner:

$$\mathbf{M}'_i = v\mathbf{M}_i(\alpha = 1) + (1-v)\mathbf{M}_i(\alpha = 0),\ i = 1, 2, 3. \tag{16}$$

This POVM essentially describes the results of an experiment in which the probability distribution $\langle \mathbf{M}'_i \rangle$ is a weighted mean of results obtained in a pure interference measurement (weight v) and a pure path measurement (weight $(1-v)$). It is obvious that also this experiment can be done easily in the optical case by replacing the semitransparant mirror B_3 in fig. 1 by a rotating beam chopper that is perfectly reflecting if the photon beam is blocked by it.

All experiments discussed till now are socalled *trivial* joint measurements because they do not yield excess information on the state of the incoming object over the one obtained by combining the results of the two pure measurements. This is the case if and only if the elements $\mathbf{M}_{ij}$ of the POVM $\{\mathbf{M}_{ij}\}$ can he written as linear combinations of the elements of the PVMs $\{\mathbf{Q}_1, \mathbf{Q}_2\}$ and $\{\mathbf{P}_l, \mathbf{P}_r\}$:

$$\mathbf{M}_{ij} = \sum_k c_{kij}\mathbf{Q}_k + \sum_n d_{nij}\mathbf{P}_n, \tag{17}$$

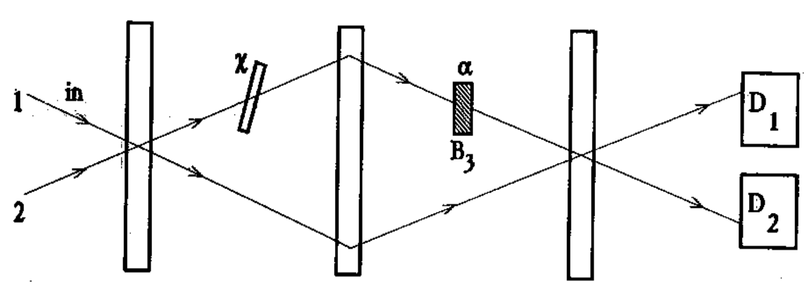

Figure 2: Trivial joint measurement of neutron interference and path. Neutron interferometer; D_i = neutron detectors; α = transmission probability of absorber B_3; χ = phase shift.

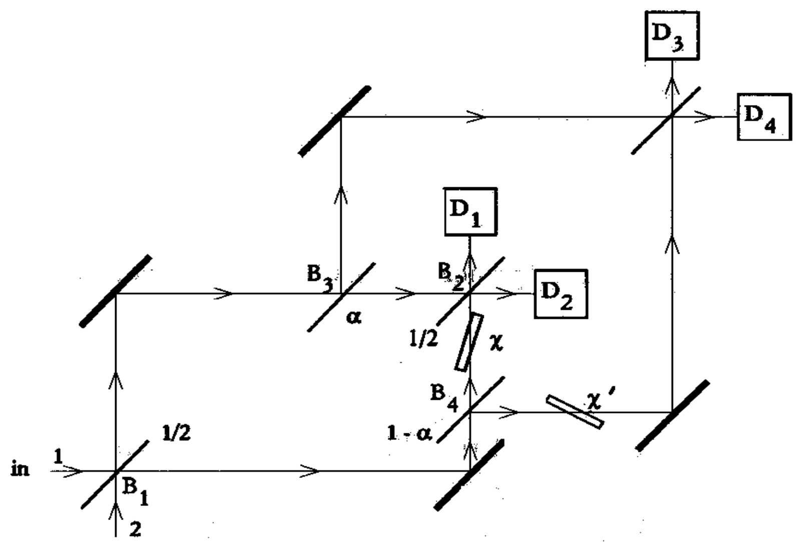

Figure 3: Nontrivial joint measurement of optical interference and path. Transparancies of beam splitters as indicated; χ, χ' = phase shifts.

because in this case the joint probability distribution $\langle \mathbf{M}_{ij} \rangle$ can be calculated from the measurement results of the pure path and interference measurements.

By Busch [6] an optical experiment is proposed that is nontrivial in this respect (cf. fig. 3). Taking transparancies and phase shifts as indicated in the figure the detection probabilities of detectors $\mathrm{D}_i,\ i = 1, \ldots, 4$, are represented by the following POVM $\{\mathbf{R}_i\}$:

$$\begin{array}{l} \mathbf{R}_1 = \frac{1}{2}[(1-\alpha)\mathbf{P}_l + \alpha\mathbf{P}_r + \sqrt{\alpha(1-\alpha)}(\mathbf{Q}_1 - \mathbf{Q}_2)], \\ \mathbf{R}_2 = \frac{1}{2}[(1-\alpha)\mathbf{P}_l + \alpha\mathbf{P}_r - \sqrt{\alpha(1-\alpha)}(\mathbf{Q}_1 - \mathbf{Q}_2)], \\ \mathbf{R}_3 = \frac{1}{2}[\alpha\mathbf{P}_l + (1-\alpha)\mathbf{P}_r + \sqrt{\alpha(1-\alpha)}(\mathbf{Q}'_1 - \mathbf{Q}'_2)], \\ \mathbf{R}_4 = \frac{1}{2}[\alpha\mathbf{P}_l + (1-\alpha)\mathbf{P}_r - \sqrt{\alpha(1-\alpha)}(\mathbf{Q}'_1 - \mathbf{Q}'_2)]. \end{array} \tag{18}$$

Here $\mathbf{Q}'_1$ and $\mathbf{Q}'_2$ are defined by substituting in (5) χ' for χ. Arranging this POVM as a bivariate POVM according to

$$(\mathbf{M}_{ij}) = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ \mathbf{R}_3 & \mathbf{R}_4 \end{pmatrix}, \tag{19}$$

we obtain for the marginals

$$\begin{array}{l} \begin{pmatrix} \mathbf{R}_1 + \mathbf{R}_2 \\ \mathbf{R}_3 + \mathbf{R}_4 \end{pmatrix} = \begin{pmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} \mathbf{P}_l \\ \mathbf{P}_r \end{pmatrix}, \\ \begin{pmatrix} \mathbf{R}_1 + \mathbf{R}_2 \\ \mathbf{R}_3 + \mathbf{R}_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \sqrt{\alpha(1-\alpha)}\cos\frac{\chi-\chi'}{2} & \frac{1}{2} - \sqrt{\alpha(1-\alpha)}\cos\frac{\chi-\chi'}{2} \\ \frac{1}{2} - \sqrt{\alpha(1-\alpha)}\cos\frac{\chi-\chi'}{2} & \frac{1}{2} + \sqrt{\alpha(1-\alpha)}\cos\frac{\chi-\chi'}{2} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_1\left(\frac{\chi+\chi'}{2}\right) \\ \mathbf{Q}_2\left(\frac{\chi+\chi'}{2}\right) \end{pmatrix}, \end{array} \tag{20}$$

indicating that the measurement can be interpreted as a joint measurement of path and interference with phase shift $(\chi + \chi')/2$.

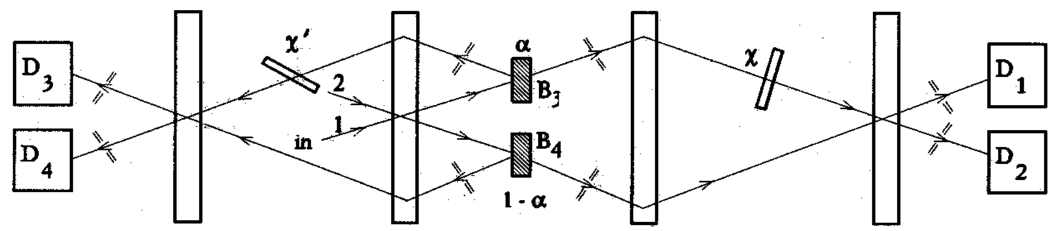

Figure 4: Nontrivial joint measurement of neutron interference and path. Transmission probabilities of scatterers B_3 and B_4 as indicated; χ, χ' = phase shifts.

An analogous neutron interference experiment is schematically given in fig. 4. We need a four-slab interferometer instead of the three-slab one that is usually employed. The elements indicated by α and $1-\alpha$, respectively, are scatterers for which the differential cross-sections in the indicated directions are such that they act like the optical beam splitters of transparancies α and $1-\alpha$. Calculation of the POVM describing this experiment demonstrates the complete analogy with the optical experiment of fig. 3, the POVM being given essentially by (18).

Inverting the nonideality matrices exhibited in (20) (which is possible if $\alpha \neq 1/2$ and $|\chi - \chi'| \neq \pi$), we obtain from (3) the Wigner measure for these experiments:

$$\begin{aligned}
\mathbf{W}_{11} &= \tfrac{1}{2}\mathbf{P}_l + \frac{(\mathbf{Q}_1' - \mathbf{Q}_2')}{4(1-2\alpha)\cos\frac{\chi-\chi'}{2}} - \frac{\alpha\left(\mathbf{Q}_1\left(\frac{\chi+\chi'}{2}\right) - \mathbf{Q}_2\left(\frac{\chi+\chi'}{2}\right)\right)}{2(1-2\alpha)},\\
\mathbf{W}_{12} &= \tfrac{1}{2}\mathbf{P}_l - \frac{(\mathbf{Q}_1' - \mathbf{Q}_2')}{4(1-2\alpha)\cos\frac{\chi-\chi'}{2}} + \frac{\alpha\left(\mathbf{Q}_1\left(\frac{\chi+\chi'}{2}\right) - \mathbf{Q}_2\left(\frac{\chi+\chi'}{2}\right)\right)}{2(1-2\alpha)},\\
\mathbf{W}_{21} &= \tfrac{1}{2}\mathbf{P}_r + \frac{(\mathbf{Q}_1 - \mathbf{Q}_2)}{4(1-2\alpha)\cos\frac{\chi-\chi'}{2}} - \frac{\alpha\left(\mathbf{Q}_1\left(\frac{\chi+\chi'}{2}\right) - \mathbf{Q}_2\left(\frac{\chi+\chi'}{2}\right)\right)}{2(1-2\alpha)},\\
\mathbf{W}_{22} &= \tfrac{1}{2}\mathbf{P}_r - \frac{(\mathbf{Q}_1 - \mathbf{Q}_2)}{4(1-2\alpha)\cos\frac{\chi-\chi'}{2}} + \frac{\alpha\left(\mathbf{Q}_1\left(\frac{\chi+\chi'}{2}\right) - \mathbf{Q}_2\left(\frac{\chi+\chi'}{2}\right)\right)}{2(1-2\alpha)}.
\end{aligned} \tag{21}$$

The marginals of this Wigner measure can easily be found as the PVMs $\{\mathbf{P}_l, \mathbf{P}_r\}$ and $\{\mathbf{Q}_1(\chi + \chi')/2, \mathbf{Q}_2(\chi+\chi')/2\}$. The nontriviality of this measurement is reflected in the fact that (21) is not of the general form (17) (as is (13)). For this reason the experiments of figs. 3 and 4 have a larger capacity of discriminating between different incoming states than the experiments of figs. 1 and 2.

References

[1] E.B. Davies, Quantum Theory of Open Systems (Academic Press, London, 1976).

[2] H. Martens and W. de Muynck, Found. of Phys. 20 (1990) 255, 357.

[3] W.W. Stoffels, Internal Report, Eindhoven University of Technology, Nov. 1990 (unpublished).

[4] J. Summhammer, H. Rauch, and D. Tuppinger, Phys. Rev. A 36 (1987) 4447.

[5] W.M. de Muynck and H. Martens, Phys. Rev. A 42 (1990) 5079.

[6] P. Busch, Found. of Phys. 17 (1987) 905.