A DYNAMICAL APPROACH TO TIME DILATION AND LENGTH CONTRACTION

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Simple models of length and time measuring instruments are studied in order to see under what conditions a relativistic description of the dynamics of accelerated motion can be consistent with the kinematic prescriptions of Lorentz contraction and time dilation. The outcomes obtained for the measuring rod are compared with a thought experiment proposed by Dewan and Beran.

Key words: relativity, time dilation, length contraction, measurement, slow transport.

1. INTRODUCTION

According to the conventional interpretation of the axioms of the special theory of relativity, length contraction and time dilation are purely kinematical. As they arise from coordinate transformations between inertial frames, they are a matter of definition. It is not sensible to ask for their cause; length contraction and time dilation are not "effects". They cannot be measured directly, because the measuring rod used to compare the lengths of an object in two different states of motion would have to be accelerated from one inertial frame to another, and would experience the same Lorentz contraction as the object (an analogous remark holds for time dilation).

The universality of the unobservability of Lorentz contraction and time dilation raises the question of the compatibility of the kinematics governed by the relativistic axioms and the dynamics of the measurements. It is generally taken for granted that measuring instruments are available answering the kinematic prescriptions inherent in the axioms of the theory, so measuring length and time is thought to be unproblematic. However, this is far from self-evident. When an object is accelerated, forces must be exerted. One may wonder whether the acceleration is independent of the way these forces are applied [1], and whether the dynamic behavior of a physical object that is used as a measuring rod or as a clock is such that, after being accelerated, its final state has the correct Lorentz contraction c.q. time dilation.

According to Born [2] a measuring rod must be a rigid body, all elements of which being contracted in the same way. Although this is an important condition for a consistent definition of *relativistic* rigidity, it does not guarantee that objects satisfying this requirement really exist. This was realized already by Laue [3] who pointed out that, like in classical mechanics, in special relativity a rigid body can be considered as the limiting case of a deformable body. Hence, with a theory of elasticity one may define which bodies are (dynamically) more rigid than others. In the present paper we follow essentially this latter idea. We shall study simple models of length and time measuring instruments in order to see under what conditions a relativistic description of the dynamics can be consistent with the kinematic prescriptions of relativistic measurement. A relativistic description of accelerated measuring instruments should show how length contraction and time dilation are brought about, thus checking by means of slow transport of clocks and measuring rods the synchronization of clocks and the comparison of lengths of measuring rods in different Lorentz frames. For a more detailed discussion of the implications our results may have on the interpretation of relativity theory the reader is referred to de Muynck [4].

We shall base our measuring instruments on a system described in a thought experiment by Dewan and Beran [5]. They consider two identical rockets, connected by a fragile thread, initially not moving with respect to an inertial frame of reference Σ. Both rockets are started simultaneously in Σ. They are assumed to have identical acceleration programs in this reference frame, thus implying the distance between the rockets, as observed in Σ, to remain constant. On the other hand the thread should be Lorentz contracted. Dewan and Beran conclude that the Lorentz contraction must cause measurable stresses "until for high enough velocities the thread finally reaches its elastic limit and breaks" (also Bell [6], who emphasizes the instructional merits of the problem).

Of course, such an object would not be a very reliable measuring instrument for measuring relativistic length. In order to get a better model of a measuring rod we shall in the following not consider equal accelerations but equal external forces exerted on the rockets. We shall also take into account the influence of the stresses on the motion of the rockets (this influence being completely suppressed by the assumption of equal accelerations in the thought experiment). Although the reasoning leading to the conclusion of the thread's breaking is correct, it seems to us that the dynamical aspects of the motion are obscured by the assumption of equal accelerations. By this assumption the problem is dealt with essentially in a kinematic way, and may correspond to a physical situation different from an accelerated measuring rod. A complete understanding can be gained only by a dynamical treatment of the motion, taking into account the dynamics of the thread. In order to have a mathematically tractable problem we have to consider a simplified model of the thread.

Our dynamical calculations give a new view on the outcome of the thought experiment, specifying conditions under which the thread does *not* break, thus providing insight into the possibility of using such an object as a measuring rod. We describe the thought experiment in Sec. 2, deriving a number of results to be used in the following sections. Models of an accelerated measuring rod and an accelerated clock are studied in Sec. 3 and Sec. 4. In Sec. 5 some conclusions are given.

2. THE DEWAN-BERAN THOUGHT EXPERIMENT

In this section the influence of the thread on the motion of the rockets is neglected. The two rockets are treated as point particles having equal and constant rest masses m. Then equal acceleration programs for the two particles in Σ can be achieved by applying equal and constant forces F to the particles, as seen in their momentarily comoving reference frames (MCRF). According to standard relativistic mechanics (see, e.g., Møller [7]) the equation of motion of each particle in Σ is

$$m\frac{d^2x}{d\tau^2} = \frac{F}{(1-v^2/c^2)^{1/2}}, \tag{1}$$

τ being the proper time of the particle, defined by

$$d\tau/dt = (1-v^2/c^2)^{1/2} = 1/\gamma. \tag{2}$$

Defining $a = F/m$ and $\theta = c/a$, we find the well–known solution

$$\begin{gathered} v(t) = at/(1+t^2/\theta^2)^{1/2}, \\ x(t) = x(0) + \lambda\left[(1+t^2/\theta^2)^{1/2} - 1\right], \\ \lambda = c^2/a. \end{gathered} \tag{3}$$

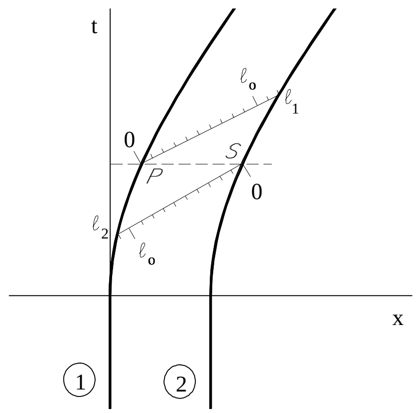

Fig. 1. Length measurements in MCRF's at equal proper times.

Taking $x_1(0) = 0,\ x_2(0) = \ell_0$, we find

$$x_2(t) = x_1(t) + \ell_0, \qquad v_2(t) = v_1(t), \tag{4}$$

corroborating the absence in Σ of Lorentz contraction of the distance between the particles.

Integrating Eq. (2) with use of Eq. (3), we find

$$\tau_1(t) = \tau_2(t) = \theta \sinh^{-1}(t/\theta).$$

Assuming the force F to be applied only during a finite time T, the particles will become stationary in the same inertial frame Σ', moving with velocity $v(T)$ with respect to Σ. In Σ' the distance between the particles, after they have both become stationary, equals $\gamma(T)\ell_0$. Since Σ' is now the rest frame of the thread, it will be stretched past its breaking limit if T is large enough.

It is interesting to observe the stretching of the thread in the MCRF of each of the particles (note that the particles have no MCRF in common!). In Fig. 1 the two world lines of the particles (1 and 2) are given as seen in Σ. At equal proper times ($\tau_2(\mathcal{S}) = \tau_1(\mathcal{P})$, corresponding to the same value of t) the distance between the particles is measured in each MCRF. The measuring rods lie along the lines of simultaneity of the MCRF's. Since the velocities in Σ are equal at equal times, the lines of simultaneity in $\mathcal{S}$ and $\mathcal{P}$ have the same direction. From this it is easily seen that the measured distances differ: $\ell_1 > \ell_2$. Straightforward calculation by means of Lorentz transformation with velocity $v = v_1 = v_2$ yields

$$\ell_1 = \ell_0\gamma + [\lambda^2 + {\ell_0}^2(\gamma^2 - 1)]^{1/2} - \lambda,$$

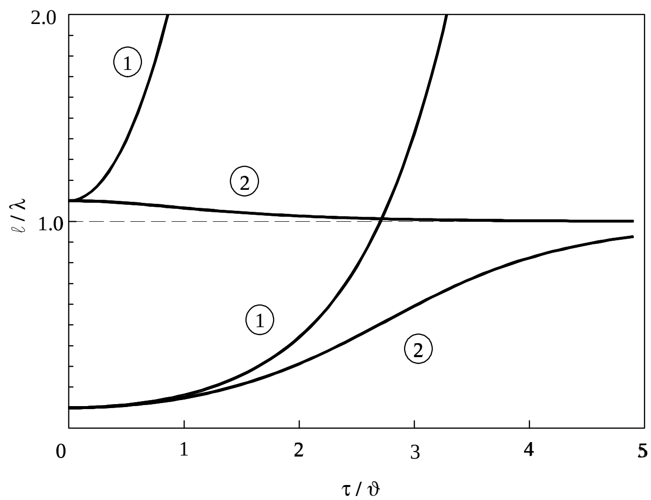

Fig. 2. Dependence on proper time of the distance between the particles, as measured in the two MCRF's, for $\ell_0 = 0.1\lambda$ and $\ell_0 = 1.1\lambda$, $\lambda = c^2/a$.

$$\ell_2 = \begin{cases} \lambda - [\lambda^2 + {\ell_0}^2(\gamma^2 - 1)]^{1/2} + \ell_0\gamma, & \text{for } \ell_0 \leq \lambda, \\ \lambda + (\ell_0 - \lambda)/\gamma, & \text{for } \ell_0 > \lambda. \end{cases} \tag{5}$$

The two cases for ℓ_2 distinguish between whether particle 2's line of simultaneity intersects world line 1 before or after the latter started off.

Figure 2 shows ℓ_1 and ℓ_2 as functions of their respective proper times τ_1 and τ_2 for two values of ℓ_0. It is clear that the distances in the MCRF's become very different for large values of τ, ℓ_1 increasing indefinitely while ℓ_2 converges to λ. We shall not deal here with the qualitatively different behavior of ℓ_2 for small and large values of ℓ_0, the latter one hardly suggesting the breaking of the thread. Desloge and Philpott [8] see the distance λ as a horizon, similar to the event horizon of a black hole. Indeed, if $\ell_0 \geq \lambda$ light emitted by particle 1 at $t \geq 0$ never reaches particle 2. Taylor and French [9] assert that λ is the limitation of proper length of an accelerating object, a being the acceleration of the front end. It is clear that it does not make sense to apply our intuitions about the behavior of threads or measuring rods to such extended objects, on the basis of considerations that are mainly kinematical. Only a field theoretic treatment, taking into account the retardation of the interaction, would be appropriate for a fully relativistic description of such objects. This will not be considered here. We shall in the following restrict ourselves to distances satisfying ℓ_0, $\ell_0\gamma \ll \lambda$, so as to make a dynamical treatment possible without the need to apply field theory. For these values it follows from (5) that the measured distances in the two MCRF's are approximately equal: $\ell_1 \approx \ell_2 \approx \ell_0\gamma$, as is also obvious in Fig. 2. It is possible, for $0 \le \ell_0 \le \lambda$, to derive from (5) the rigorous inequality

$$|\ell_1 - \ell_2| \le \ell_1\ell_2/\lambda, \tag{6}$$

showing the relative difference to be small as long as the distance remains much smaller than λ. Only if this condition is fulfilled is it possible to define a MCRF in which the thread as a whole is approximately stationary, thus making it possible to ignore the internal dynamics of the thread.

3. A RELATIVISTIC MEASURING ROD

If we replace the thread connecting the accelerated particles in the Dewan-Beran thought experiment by an ideal massless spring, we obtain a simple model of a relativistic measuring rod. In the dynamical treatment both the external forces and the spring determine the motion of the particles. In order to be able to define a spring, we restrict ourselves to small relative velocities ($\ll c$) and a small distance ($\ll \lambda = c^2/a$) between the particles. This implies a radical difference from the kinematical approach, since in the limit of vanishing spring constant these restrictions cannot be satisfied: the spring is essential to hold the particles together. Under these conditions it is possible to define a MCRF with respect to which both particles are approximately at rest. In this MCRF, the ideal spring connecting the particles behaves according to Hooke's law.

The assumption of small distance, $\ell\gamma \ll \lambda$, allows us to write $\ell_1 \approx \ell_2 \approx \ell\gamma$. Here $\ell = x_2 - x_1$ and $\gamma = \gamma(v)$, with v the velocity of the MCRF. If ℓ_0 is the equilibrium length and k the spring constant, then the spring exerts a force $k\,(\ell\gamma - \ell_0)$ in opposite directions for the two particles. As in Sec. 2 we assume constant and equal external forces F to be exerted on both particles. The relativistic equations of motion can then be written as

$$m\frac{dv_1}{dt} = \left(1 - {v_1}^2/c^2\right)^{3/2}[F - k(\ell_0 - \ell\gamma)], \tag{7}$$

$$m\frac{dv_2}{dt} = \left(1 - {v_2}^2/c^2\right)^{3/2}[F + k(\ell_0 - \ell\gamma)]. \tag{8}$$

As we see by adding Eq. (7) and Eq. (8), the restriction to small relative velocities ($v_1 \approx v_2$) implies that v is equal to the velocity of the unconnected particles, as given by Eq. (3). The proper time τ of the MCRF is defined by $d\tau/dt = 1/\gamma$, $\tau(0) = 0$.

Since the relative movement of the particles is of particular interest we define $w(t) = v_2(t) - v_1(t) = d\ell/dt$. We find a differential equation for $w(t)$ by subtracting Eq. (7) from Eq. (8):

$$\begin{aligned} m\frac{dw}{dt} &= F\left[(1-{v_2}^2/c^2)^{3/2} - (1-{v_1}^2/c^2)^{3/2}\right] \\ &+ k\,(\ell_0 - \ell\gamma)\left[(1-{v_2}^2/c^2)^{3/2} + (1-{v_1}^2/c^2)^{3/2}\right]. \end{aligned} \tag{9}$$

With the assumption $|w| \ll |v|$, we approximate

$$(1-{v_2}^2/c^2)^{3/2} + (1-{v_1}^2/c^2)^{3/2} \approx 2/\gamma^3,$$

$$(1-{v_2}^2/c^2)^{3/2} - (1-{v_1}^2/c^2)^{3/2} \approx -3vw/(\gamma c^2).$$

We now have the following approximation for Eq. (9):

$$m\frac{dw}{dt} \approx F\left(-\frac{3vw}{\gamma c^2}\right) + 2k\frac{\ell_0 - \ell\gamma}{\gamma^3}. \tag{10}$$

From the definition of γ and v we have $dv/dt = a/\gamma^3$ and $d\gamma/dt = av/c^2$. We rewrite by use of the chain rule:

$$\frac{d^2\ell\gamma}{d\tau^2} = \gamma\frac{d}{dt}\left(\gamma\frac{d\ell\gamma}{dt}\right) = 3\gamma^2\frac{a}{c^2}vw + \gamma^3\frac{dw}{dt} + \frac{a^2\ell\gamma}{c^2}. \tag{11}$$

Insertion of Eq. (10) into Eq. (11) gives

$$\frac{d^2\ell\gamma}{d\tau^2} \approx -\left(2k/m - a^2/c^2\right)\left(\ell\gamma - \frac{2k/m}{2k/m - a^2/c^2}\ell_0\right). \tag{12}$$

The solution is now simple:

$$\ell\gamma = \frac{2k/m}{2k/m - a^2/c^2}\ell_0 + A\sin\omega\tau + B\cos\omega\tau, \tag{13}$$

with $\omega = (2k/m - a^2/c^2)^{1/2}$. With $d(\ell\gamma)/d\tau = 0$ at $\tau = 0$ we find $A = 0$. B is determined by ℓ_0. After some rewriting we obtain[1]

$$\ell\gamma = \ell_0[1 + \frac{a^2}{\omega^2 c^2}(1 - \cos\omega\tau)]. \tag{14}$$

[1] It can be shown that this result can be generalized for nonidentical forces F_1 and F_2 and masses m_1 and m_2 satisfying $F_1/m_1 = F_2/m_2 = a$. The generalization merely involves replacing m in Eqs. (7–14) by the reduced mass μ.

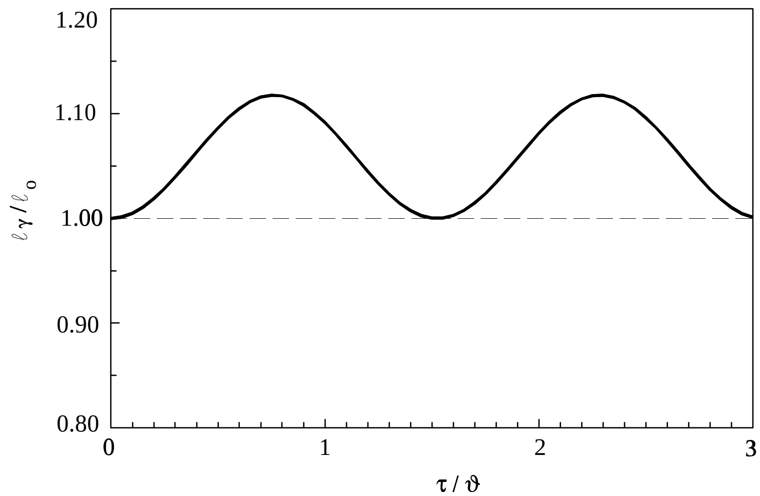

Fig. 3. The behavior of the measuring rod as seen from the MCRF: $\ell\gamma/\ell_0$ vs. τ/θ (with $kc^2/(ma^2) = 9$; the broken curve represents an ideal rod).

Viewed from successive MCRF's the distance between the particles oscillates between the initial length and a length at which the spring is expanded maximally. The oscillations are harmonic in τ (cf. Fig. 3). As we take k larger, B vanishes and the length around which $\ell\gamma$ oscillates approaches ℓ_0. In Σ we see oscillations with respect to the Lorentz contracted length (cf. Fig. 4), the frequency increasing and the amplitude decreasing if k is chosen to be larger. By increasing k, we can let the system behave arbitrarily closely like an ideal measuring rod, which has $\ell\gamma = \text{const}$, though for finite k it will never be one. For Figs. 3 and 4 we used $\lambda = 9\times 10^8\,\text{m}$, $\theta = 3\,\text{s}$ and $\ell_0 = 1\,\text{m}$. A numerical solution using an extended Runge-Kutta-Fehlberg algorithm [10] to integrate exactly Eqs. (7)–(8) is in good agreement with these results.

From Eq. (14) it is clear that the assumption $\ell\gamma \ll \lambda$ implies $a^2\ell_0/(\omega^2c^2) \ll \lambda$. By the definition of ω, we have

$$2k/m - a^2/c^2 \gg a^2\ell_0/(\lambda c^2). \tag{15}$$

Thus there is a lower bound for k: the equations which we use to describe the system of the connected particles are indeed not valid in the limit of an extremely weak spring. The discontinuity between $k = 0$ and $k = k_{\text{min}}$ marks the fundamental difference between the kinematical and the dynamical approach.

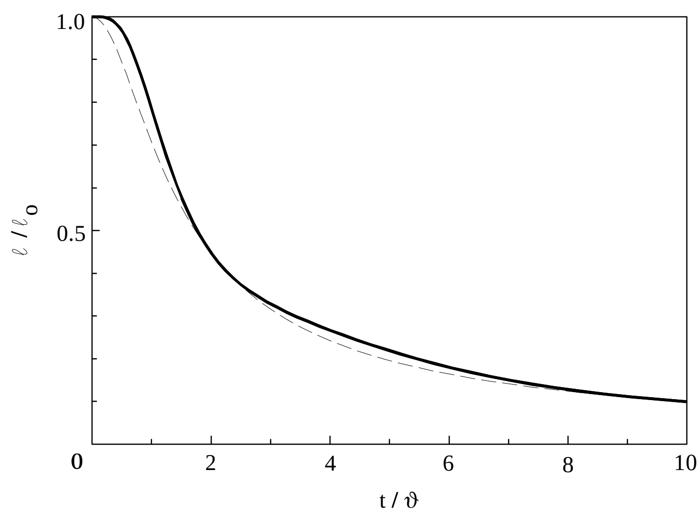

Fig. 4. The behavior of the measuring rod as seen by an observer in Σ: ℓ/ℓ_0 vs. t/θ, for $kc^2/(ma^2) = 9$. The broken curve shows the behavior of an ideal rod, $\ell/\ell_0 = 1/\gamma$.

4. A RELATIVISTIC CLOCK

Our treatment of an accelerated measuring rod shows how the Lorentz contraction is brought about dynamically. With the same approach we shall investigate the behavior of a clock under acceleration. At rest, our model of a clock consists of a particle p turning circles around a center c, a fixed point in space. An ideal massless spring connects the particle to the center. The completion of a loop around the center corresponds to a tick of the clock. Acceleration of the clock is performed by a constant force both on the particle and on the center; for simplicity, we prescribe the motion of the center. The reduced mass of the system is equal to the mass m of the particle. Classically, the period T of one loop is independent of size and shape of the orbit. It is well known that we have $T = 2\pi(m/k)^{1/2}$ in all cases: at rest, in uniform motion, and under acceleration.

For the relativistic treatment, we define some new symbols: the position coordinates $(x_{\rm p}, y_{\rm p})$ for the position of the particle and $v_{\rm p}$ for its velocity, $v_{\rm p}^2 = (dx_{\rm p}/dt)^2 + (dy_{\rm p}/dt)^2$. We use $x_{\rm c}$ for the x-position of the center. The spring is ideal and has zero equilibrium length ℓ_0. It is assumed that in the center's MCRF, the particle has a low velocity and $\tau_{\rm p} \approx \tau_{\rm c}$. We first consider an inertial clock. We let the particle describe a circular orbit in the center's MCRF (coordinates $\hat{x}, \hat{t}$):

$$\hat{x}_{\rm p} = r\cos\omega\hat{t}, \qquad \hat{y}_{\rm p} = r\sin\omega\hat{t}, \qquad \omega = (k/m)^{1/2}. \tag{16}$$

We have $\hat{T} = 2\pi(m/k)^{1/2}$. If the MCRF has velocity V with respect to Σ, then Lorentz transforming to Σ, $\hat{x} = \gamma\,(x - Vt)$, $\hat{y} = y$, $\hat{t} = \gamma\,(t - xV/c^2)$, we obtain

$$x_{\rm p} \approx (r/\gamma)\cos\omega(t/\gamma) + Vt, \qquad y_{\rm p} \approx r\sin\omega(t/\gamma), \tag{17}$$

where we approximated $\tau_{\rm p} \approx \tau_{\rm c} \approx t/\gamma$. The particle describes a Lorentz contracted circle, an ellipse with ratio γ between its principal axes. Note that we have $T = \hat{T}\gamma$, which is just the time dilation.

Now we accelerate the clock. The center follows a prescribed hyperbola in space-time in accordance with Eq. (3). We use the velocity $v_{\rm c}$ of the center to define $\gamma_{\rm c} = \gamma(v_{\rm c})$. The equations of motion are

$$\frac{d^2x_{\rm p}}{d\tau_{\rm p}^2} = \gamma_{\rm p}[a - (k/m)\ell\gamma_{\rm p}], \tag{18}$$

$$\frac{d^2y_{\rm p}}{d\tau_{\rm p}^2} = -(k/m)y_{\rm p}, \tag{19}$$

in which we redefined $\ell = x_{\rm p} - x_{\rm c}$; we use $\tau_{\rm p}$ because the forces are exerted on the particle.

We solve the equations of motion in the same way as in the previous section. Equating, as before, $\tau_{\rm p}$ and $\tau_{\rm c}$ we obtain for the oscillation in the x-direction

$$\ell\gamma = r\cos\omega_x\tau_{\rm c}, \qquad \omega_x = (k/m - a^2/c^2)^{1/2}. \tag{20}$$

For $y_{\rm p}$ the solution is simply

$$y_{\rm p} = r\sin\omega_y\tau_{\rm c}, \qquad \omega_y = (k/m)^{1/2}. \tag{21}$$

The oscillation in the y-direction is a classical harmonic oscillation in the proper time $\tau_{\rm c}$. Since only the vertical oscillation is important for the functioning of the clock, the time dilation is followed ideally: there are constant amounts of *proper* time between the ticks.

5. CONCLUSIONS

We studied a simple model of an accelerated relativistic measuring rod and clock, viz., two particles connected by a spring in either linear or circular motion. It was investigated whether a relativistic dynamical description of these models yields results that are consistent with the kinematical prescription of relativity theory stipulating

Lorentz contraction and time dilation. It was found that the measuring rod satisfies the Lorentz contraction only in the limit of an infinitely large spring constant, a finite spring constant yielding deviations from ideal behavior. A tentative interpretation is that only ideal measuring instruments must satisfy the kinematical prescriptions, and that a spring having an infinitely large spring constant is a model of a rigid body meeting this requirement. Because the reading of a clock can be realized by considering transverse motion (which is less sensitive to relativistic effects than the longitudinal one), it turns out to be easier to have an ideal relativistic clock, reproducing time dilation exactly for arbitrary finite spring constant.

The model sheds new light on a thought experiment proposed by Dewan and Beran [5], considering two rockets (particles) connected by a thread. By applying equal and constant external forces instead of accelerations we are able to take into account more fully the relativistic dynamics of the system. Our results lead to the following conclusions:

1. If the external force is large ($a = F/m \gtrsim c^2/\ell_0$), defining a connecting thread is problematic. A description of the interaction between the particles would involve field theory.

2. If the acceleration is sufficiently gentle, the amplitude of the length oscillation can be made so small that the thread will not break. The particles are then *kept together* by the thread. This physical possibility does not obtain in the Dewan-Beran thought experiment because of the latter's kinematical character.

3. If the acceleration is intermediate, the maximum increase in length may become larger than the thread can take, and the thread will break as in the Dewan-Beran thought experiment.

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