Preparation and Measurement: Two Independent Sources of Uncertainty in Quantum Mechanics

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In the Copenhagen interpretation the Heisenberg inequality $\Delta Q \Delta P \geq \hbar/2$ is interpreted as the mathematical expression of the concept of complementarity, quantifying the mutual disturbance necessarily taking place in a simultaneous or joint measurement of incompatible observables. This interpretation was criticized a long time ago, and has recently been challenged in an experimental way. These criticisms can be substantiated by using the generalized formalism of positive operator-valued measures, from which an inequality, different from the Heisenberg inequality, can be derived, precisely illustrating the Copenhagen concept of complementarity. The different roles of preparation and measurement in creating uncertainty in quantum mechanics are discussed.

1 INTRODUCTION

The Copenhagen view on the meaning of quantum mechanics originated largely from the consideration of so-called "thought experiments," like the double-slit experiment and the γ microscope. These experiments demonstrate that there is a mutual disturbance of the measurement results in a joint measurement of two incompatible observables A and B (like position Q and momentum P). The Heisenberg uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} \mid \langle [A, B]_- \rangle \mid, \tag{1}$$

in which ΔA and ΔB are standard deviations, has often been interpreted as the mathematical expression of this disturbance (in Heisenberg's paper [1] only position Q and momentum P are considered). However, as noted by Ballentine [2], this uncertainty relation does not seem to have any bearing on the issue of joint measurement, because it can be experimentally tested by measuring each of the observables separately. Moreover, such an interpretation is at variance with the standard formalism developed by Dirac and von Neumann, which allows the joint

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measurement of only *compatible* observables. According to Ballentine quantum mechanics is silent about the joint measurement of incompatible observables. If this were true, however, what would this mean for the disturbance idea originating from the "thought experiments"? How could these experiments be useful in clarifying the meaning of a mathematical formalism that is not capable of yielding a description of such experiments?

Nowadays measurements like the double-slit experiment no longer are "thought" experiments [3, 4, 5, 6, 7, 8, 9, 10], and complementarity, in the sense of mutual disturbance, has been experimentally demonstrated in an unequivocal way. However, in agreement with Ballentine's observation the relation of these experiments with the Heisenberg inequality (1) has proved controversial [11, 12]. Whereas Storey et al. [11] conclude that "the principle of complementarity is a consequence of the Heisenberg uncertainty relation," Scully et al. [12] observe that "The principle of complementarity is manifest although the position-momentum uncertainty relation plays no role." Duerr et al. [10] stress that quantum correlations due to the interaction of object and detector, rather than "classical" momentum transfer, enforces the loss of interference in a which-way measurement. In their experiment momentum disturbance is not large enough to account for the loss of interference if the measurement arrangement is changed so as to yield "which-way" information.

Actually, two questions are at stake here. First, the question might be posed whether the Heisenberg inequality of *position and momentum* is the relevant one for interference experiments. Second, there is the problem observed by Ballentine, which is the more fundamental question whether the Heisenberg inequality is applicable at all. Contrary to the latter question, the former might be thought to have a relatively simple answer. In general, interference experiments like the one of Duerr et al. [10] are not joint measurements of position and momentum but of a different pair of observables A and B (see sect. 4 for an example). Hence, rather than the inequality $\Delta Q \Delta P \geq \hbar/2$, relation (1) with observables A, B *different from* P, Q seems to be relevant to the experiment. Although position and momentum may also be disturbed by the interaction with the detector, this need not be related to complementarity because A and B rather than Q and P are involved in the correlations between object and detector. Hence, the controversy could be resolved by pointing out which (incompatible) observables are measured jointly in the experiment. However, then we would still have to deal with quantum mechanics' alleged silence with respect to joint measurements.

The second problem, viz. whether (1) is applicable at all to the joint measurement of incompatible observables A and B, has more far-reaching consequences because it points to a fundamental confusion regarding complementarity within the Copenhagen interpretation. This is due to the poor distinction made between the different aspects of *preparation* and *measurement* involved in physical experimentation. As a matter of fact, in the Copenhagen interpretation a measurement is not perceived as a means of obtaining information about the *initial* (premeasurement) state of the object, but as a way of preparing the object in some *final* (postmeasurement) state. Due to this view on the meaning of "measurement" there is insufficient awareness that both the preparation of the initial state and the measurement may contribute to the dispersion of an observable. The Copenhagen issue of complementarity actually has two aspects, viz. the aspects of preparation and measurement, which are not distinguished clearly enough. If such a distinction is duly made, it is not difficult to realize that the notion of "measurement disturbance" should apply to the latter aspect, whereas the Heisenberg uncertainty relation refers to the former. With no proper distinction between preparation and measurement the Copenhagen interpretation was bound to amalgamate the two forms of complementarity, thus interpreting the Heisenberg uncertainty relation as a property of (joint) measurement.

In sect. 2 the origin of the Copenhagen confusion of preparation and measurement is discussed first. The purpose of the present paper is to demonstrate that this confusion is largely a consequence of the inadequateness of the standard formalism for the purpose of yielding a description of certain quantum mechanical experiments, and joint measurements of incompatible observables in particular. To describe such measurements it is necessary to generalize the quantum mechanical formalism so as to encompass positive operator-valued measures [13, 14, 15, 16] (POVMs); the standard formalism is restricted to the projection-valued measures corresponding to the spectral representations of self-adjoint operators. The generalized formalism is briefly discussed in sect. 3. In sect. 4 the generalized formalism is applied to an atomic beam interference experiment that can be seen as a realization of the double-slit experiment. By employing the generalized formalism of POVMs it is possible to interpret this experiment as a joint nonideal measurement of incompatible observables like the ones considered in the "thought experiments". An inequality, derived from the generalized theory by Martens [17], yields an adequate expression of the mutual disturbance of the information on the initial probability distributions of two incompatible observables in a joint measurement of these observables. How both contributions to complementarity can be distinguished in the measurement results obtained in such experiments is discussed in sect. 5.

2 CONFUSION OF PREPARATION AND MEASUREMENT

The confusion of preparation and measurement is already present in the Copenhagen thesis that quantum mechanics is a complete theory. As a consequence of this thesis a physical quantity cannot have a well-defined value *preceding the measurement* (because this would correspond to an "element of physical reality" as employed by Einstein, Podolsky and Rosen [18] to demonstrate the *in*completeness of quantum mechanics). For this reason a quantum mechanical measurement cannot serve to ascertain this value in the way customary in classical mechanics. Heisenberg [19] proposed an alternative for quantum mechanics, to the effect that the value of an observable is well-defined *immediately after the measurement*, and, hence, is more or less created by the measurement [20]. For Heisenberg his uncertainty relation did not refer to the past (i.e. to the initial state), but to the future (i.e. the final state): it was seen as a consequence of the disturbing influence of the measurement on observables that are incompatible with the measured one. Hence, for Heisenberg a quantum mechanical measurement was a *preparation* (of the final state of the object), rather than a determination of certain properties of the initial state. As emphasized by Ballentine, the interpretation of the Heisenberg uncertainty relation (1) usually found in quantum mechanics textbooks, is in disagreement with Heisenberg's views, because in the textbook view this relation is not considered a property of the *measurement process* but, rather, of the *initial object state*. The controversy, referred to above, on the relation between the principle of complementarity and the Heisenberg inequality stems from doubts with respect to Heisenberg's conception of measurement, in which a position (or path) measurement is thought to disturb final momentum distribution so as to wipe out interference.

Also Bohr [21, 22] did not draw a clear distinction between preparation and measurement. He always referred to the *complete* experimental arrangement (often indicated as "the measuring instrument") serving to *define* the measured observable. For Bohr the uncertainty relation (1) was an expression of our limitations in *jointly defining* complementary quantities (like position and momentum) within the context of a measurement. He did not distinguish different phases of the measurement. More particularly he did not distinguish different contributions to complementarity from the preparation of the initial state and from the disturbance by the measurement. According to Bohr the uncertainty relation refers to the "latitudes" of the definition of incompatible observables within the context of a well-defined measurement arrangement, deemed valid for the measurement as a whole. Incidentally, we see a manifest difference here from Heisenberg's views, a difference that may have confused anyone trying to understand the Copenhagen interpretation as a consistent way of looking at quantum mechanics. Moreover, the discrepancy between the Copenhagen interpretations of the uncertainty relation (viz. as a property of the *measurement*, either *during* this measurement (Bohr), or afterwards (Heisenberg)) and the textbook interpretation (viz. as a property of the *preparation preceding the measurement*) may have caused some uneasiness in many students.

Obviously, two completely different issues are at stake here, corresponding to different forms of complementarity. As stressed by Ballentine, the Heisenberg uncertainty relation (1), in which ΔA and ΔB are standard deviations in separately performed measurements, should be taken, in agreement with textbook interpretation, as referring to the *preparation of the initial state*. On the other hand, the

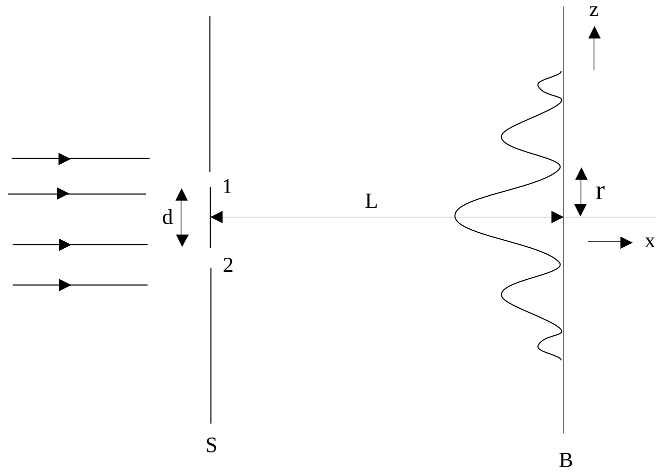

Figure 1: Double-slit experiment

Copenhagen idea of complementarity in the sense of mutual disturbance in a joint measurement of incompatible observables, is certainly not without a physical basis. Thus, in the double-slit experiment (cf. figure 1) Bohr demonstrated that, if the quantum mechanical character of screen S is taken into account, our possibility to define the position and momentum of a particle passing the slits is limited by the Heisenberg uncertainty relation

$$\Delta z_S \Delta p_{z_S} \geq \hbar/2 \tag{2}$$

of the screen observables z_S and p_{z_S}. As a matter of fact, the lower bounds of the latitudes δz and δp_z with which particle position and momentum are defined, are determined by the standard deviations Δz_S and Δp_{z_S}, respectively [23]. Hence, these latitudes must satisfy the inequality

$$\delta z \delta p_z \gtrsim \hbar/2. \tag{3}$$

In Heisenberg's terminology this inequality can be interpreted as expressing a lower bound for the disturbing influence exerted by the measuring instrument on the particle, thus causing the *post*measurement state of the object to satisfy an uncertainty relation.

Inequality (3) should be distinguished from the uncertainty relation

$$\Delta z \Delta p_z \geq \hbar/2 \tag{4}$$

satisfied by the standard deviations Δz and Δp_z of position and momentum of the particle in its *initial* state. Whereas inequality (4), being an instance of inequality (1), does not refer in any way to joint measurement of position and momentum, but can be interpreted as a property of the *preparation of the object preceding the measurement*, does inequality (3) refer to the *measurement process*, since it is derived from a relation (viz. (2)) satisfied by (a part of) the measurement arrangement (screen S).

Unfortunately, in discussions of the double-slit experiment such a distinction usually is not made. On the contrary, equating the quantities δz and δp_z from (3) with the standard deviations Δz and Δp_z, the derivation of (3) is generally interpreted as an illustration of relation (4). As a consequence it is not sufficiently realized that preparation of the initial state and (joint) measurement are two distinct physical sources of uncertainty, yielding similar but physically distinct uncertainty relations that express different forms of complementarity. Only the former one is represented by a relation (viz. (1)), that can straightforwardly be derived from the standard formalism. Bohr's analysis of the double-slit experiment demonstrates that there is a second form of complementarity, which is *not* a property of the preparation of the initial state, but which is due to mutual disturbance in a joint measurement of position and momentum.

One important cause of the mixing up of the two forms of complementarity is the fact, stressed by Ballentine, that the quantum mechanical formalism, as axiomatized by von Neumann and Dirac, defies a description of joint measurement of incompatible observables. In particular, such a measurement would have to yield joint probability distributions of the incompatible observables. However, within the standard formalism no mathematical quantities can be found that are able to play such a role. Thus, according to Wigner's theorem [24] no positive phase space distribution functions $f(q, p)$ exist that are linear functionals of the density operator ρ such that $\int dp\ f(q, p) = \langle q|\rho|q\rangle$ and $\int dq\ f(q, p) = \langle p|\rho|p\rangle$. Also von Neumann's projection postulate is often interpreted as prohibiting the joint measurement of incompatible observables, since there is no unambiguous eigenstate that can serve as the final state of such a measurement. For this reason only measurements of one single observable, for which the Heisenberg relation has an unambiguous significance, are usually considered in axiomatic treatments.

On the other hand, Ballentine's judgment with respect to the inability of the quantum mechanical formalism to deal with the second kind of complementarity seems to be too pessimistic. Thus, for specific measurement procedures *generalized* Heisenberg uncertainty relations have been derived [8, 9, 25, 26], different from the Heisenberg relation, in which the uncertainties contain contributions from both sources. Moreover, in the following it will be demonstrated that the generalized quantum mechanical formalism is able to deal with the two forms of complementarity separately, thus distinguishing the contributions due to preparation (of the initial state) and (joint) measurement.

One further possible source of confusion should be mentioned here, being related to the fact that Heisenberg's conception of measurement is not the most general one. In general not the final state of the object, but that of the measuring instrument is determining the information obtained by a measurement. Only in very special cases is the final state of the object indicative of the measurement result recorded by the measuring instrument. More generally the state of the object may be disturbed by the measurement in an arbitrary way. It is an interesting question whether this *preparatively* disturbing effect of measurement satisfies some principle of complementarity, different from the one described by the (universally satisfied) Heisenberg inequality (1) as applied to the final object state [27, 28, 29]. This preparative disturbance is not considered here, however. In the following we shall restrict to the *determinative* (or retrodictive) aspects of complementarity referring to the information about the *initial* state to be obtained by a measurement. As will be seen in the next section, the idea of mutual disturbance in a joint measurement of incompatible observables can be given a meaning in this determinative sense (often associated with measurement inaccuracy). Equating measurement disturbances in the preparative and determinative senses (as is done when identifying (3) and (4)) might be possible in very special cases in which the measurement result can be directly read off the final state of the object, but is not justified in general measurements.

3 GENERALIZED MEASUREMENTS

In the generalized quantum mechanical formalism the notion of a quantum mechanical measurement is generalized so as to encompass measurement procedures that can be interpreted as joint measurements of incompatible observables of the type considered in the "thought experiments." A possibility to do so is offered by the *operational approach* [13, 14, 15, 16], in which the interaction between object and measuring instrument is treated quantum-mechanically, and measurement results are associated with pointer positions of the latter. If ρ and ρ_a are the initial density operators of object and measuring instrument, respectively, then the probability of a measurement result is obtained as the expectation value of the spectral representation $\{E_m^{(a)}\}$ of some observable of the measuring instrument in the final state $\rho_f = U\rho\rho_a U^\dagger,\ U = exp(-iHT/\hbar)$ of the measurement. Thus, $p_m = Tr_{oa}\rho_f E_m^{(a)}$. This quantity can be interpreted as a property of the *initial* object state, $p_m = Tr_o\rho M_m$, with $M_m = Tr_a\rho_a U^\dagger E_m^{(a)} U$.

Whereas in the standard formalism quantum mechanical probabilities p_m are represented by the expectation values of mutually commuting projection operators ($p_m = \langle E_m\rangle,\ E_m^2 = E_m,\ [E_m, E_{m'}]_- = O$), the generalized formalism allows these probabilities to be represented by expectation values of operators M_m that are not necessarily projection operators, and need not commute ($M_m^2 \neq M_m,\ [M_m, M_{m'}]_- \neq O$ in general). The operators $M_m,\ O \leq M_m \leq I,\ \sum_m M_m = I$ generate a positive operator-valued measure (POVM); the observables of the standard Dirac-von Neumann formalism are restricted to those POVMs of which the elements are mutually commuting projection operators (so-called projection-valued measures).

After having generalized the notion of a quantum mechanical observable it is possible to define a relation of partial ordering between observables, expressing that the measurement represented by one POVM can be interpreted as a nonideal measurement of another [17]. Thus, for simplicity restricting to discrete spectra, we say that a POVM $\{R_m\}$ represents a *nonideal* measurement of the (generalized or standard) observable $\{M_{m'}\}$ if the following relation holds between the elements of the POVMs:

$$R_m = \sum_{m'} \lambda_{mm'} M_{m'},\ \lambda_{mm'} \geq 0,\ \sum_m \lambda_{mm'} = 1. \tag{5}$$

The matrix $(\lambda_{mm'})$ is the nonideality matrix. It is a so-called *stochastic* matrix [30]. Its elements $\lambda_{mm'}$ can be interpreted as conditional probabilities of finding measurement result a_m if an ideal measurement had yielded measurement result $a_{m'}$. In the case of an ideal measurement the nonideality matrix $(\lambda_{mm'})$ reduces to the unit matrix $(\delta_{mm'})$. As an example we mention photon counting using an inefficient photon detector (quantum efficiency $\eta < 1$), for which the probability of detecting m photons during a time interval T can be found (cf. Kelley and Kleiner [31]) as:

$$p_m(T) = Tr\rho\mathcal{N}\left(\frac{(\eta a^\dagger a)^m}{m!}\exp(-\eta a^\dagger a)\right) \tag{6}$$

(in which $a^\dagger$ and a are photon creation and annihilation operators, and $\mathcal{N}$ is the normal ordering operator). Defining the POVM $\{R_m\}$ of the inefficient measurement by means of the equality $p_m(T) = Tr\rho R_m$, it is not difficult to prove that R_m can be written in the form

$$R_m = \sum_{n=0}^{\infty} \lambda_{mn} |n\rangle\langle n|, \tag{7}$$

with $|n\rangle\langle n|$ the projection operator projecting on number state $|n\rangle$, and

$$\lambda_{mn} = \left\{ \begin{array}{cc} 0, & m > n, \\ \binom{n}{m}\eta^m(1-\eta)^{n-m}, & m \leq n. \end{array} \right. \tag{8}$$

For $\eta = 1$ the nonideality matrix is seen to reduce to the unit matrix, and the POVM (7) to coincide with the projection-valued measure corresponding to the spectral representation of the photon number observable $N = \sum_{n=0}^{\infty} n|n\rangle\langle n|$.

Nonideality relations of type (5) are well-known from the theory of transmission channels in the classical theory of stochastic processes [32], where the nonideality matrix describes the crossing of signals between subchannels. It should be noted, however, that, notwithstanding the classical origin of the latter subject, the nonideality relation (5) may be of a *quantum mechanical* nature. Thus, the interaction of the electromagnetic field with the inefficient detector is a quantum mechanical process just like the interaction with an ideal photon detector is. Relations of the type (5) are abundant in the quantum theory of measurement. They can be employed to characterize the quantum mechanical idea of mutual disturbance in a joint measurement of incompatible observables.

Generalizing the notion of quantum mechanical measurement to the joint measurement of two (generalized) observables, it seems reasonable to require that such a measurement should yield a *bivariate* joint probability distribution p_{mn}, satisfying $p_{mn} \geq 0, \sum_{mn} p_{mn} = 1$. Here m and n label the possible values of the two observables measured jointly, corresponding to pointer positions of two different pointers (one for each observable) being jointly read for each individual preparation of an object. It is assumed that, analogous to the case of single measurement, the probabilities p_{mn} of finding the pair (m, n) are represented in the formalism by the expectation values $\langle R_{mn} \rangle$ of a bivariate POVM $\{R_{mn}\},\ R_{mn} \geq O,\ \sum_{mn} R_{mn} = I$ in the initial state of the object. Then the marginal probability distributions $\{\sum_n p_{mn}\}$ and $\{\sum_m p_{mn}\}$ are expectation values of POVMs $\{M_m = \sum_n R_{mn}\}$ and $\{N_n = \sum_m R_{mn}\}$, respectively, which correspond to the (generalized) observables jointly measured.

It can be proven [33] that, if the observables corresponding to the POVMs $\{M_m\}$ and $\{N_n\}$ are standard observables (i.e. if the operators M_m and N_n are projection operators), then joint measurement is possible only if these observables commute. This result, derived here from the generalized formalism, corroborates the standard formalism for those measurements to which the latter is applicable. Note, however, that in general commutativity of the operators M_m and N_n is not a necessary condition for joint measurability of generalized observables (see sect. 4 for an example).

The notion of joint measurement can be extended in the following way. We say that a measurement, represented by a bivariate POVM $\{R_{mn}\}$, can be interpreted as a *joint nonideal* measurement of the observables $\{M_m\}$ and $\{N_n\}$ if the marginals $\{\sum_n R_{mn}\}$ and $\{\sum_m R_{mn}\}$ of the bivariate POVM $\{R_{mn}\}$, describing the joint measurement, represent *nonideal* measurements of observables $\{M_m\}$ and $\{N_n\}$. Then, in accordance with (5) two nonideality matrices $(\lambda_{mm'})$ and $(\mu_{nn'})$ should exist, such that

$$\begin{array}{l} \sum_n R_{mn} = \sum_{m'} \lambda_{mm'} M_{m'},\ \lambda_{mm'} \geq 0,\ \sum_m \lambda_{mm'} = 1, \\ \sum_m R_{mn} = \sum_{n'} \mu_{nn'} N_{n'},\ \mu_{nn'} \geq 0,\ \sum_n \mu_{nn'} = 1. \end{array} \quad (9)$$

It is possible that $\{M_m\}$ and $\{N_n\}$ are incompatible standard observables. To demonstrate that the joint measurement scheme, given above, is a useful one, an atomic beam interference experiment is discussed in the next section as an example, satisfying the definition of a joint nonideal measurement of two incompatible standard observables. It should be noted that this example is not an exceptional one, but can be supplemented by many others [34, 35, 36, 37]. For instance, in analogy "eight-port optical homodyning [5]" can be interpreted as a joint nonideal measurement of the observables $Q = (a + a^\dagger)/\sqrt{2}$ and $P = (a - a^\dagger)/i\sqrt{2}$ of a monochromatic mode of the electromagnetic field.

If $\{M_m\}$ and $\{N_n\}$ are standard observables, then the nonidealities expressed by the nonideality matrices $(\lambda_{mm'})$ and $(\mu_{nn'})$ can be proven [17] to satisfy the characteristic traits of the type of complementarity that is due to mutual disturbance in a joint measurement of incompatible observables as dealt with in the "thought experiments." A measure of the departure of a nonideality matrix from the unit matrix is required for this. A well-known quantity serving this purpose is Shannon's channel capacity [32]. Here we consider a closely related quantity, viz. the average row entropy of the nonideality matrix $(\lambda_{mm'})$,

$$J_{(\lambda)} = -\frac{1}{N} \sum_{mm'} \lambda_{mm'} \ln \frac{\lambda_{mm'}}{\sum_{m''} \lambda_{mm''}}, \tag{10}$$

that (restricting to square $N \times N$ matrices) satisfies the following properties:

$$\begin{array}{l} 0 \leq J_{(\lambda)} \leq \ln N, \\ J_{(\lambda)} = 0 \text{ if } \lambda_{mm'} = \delta_{mm'}, \\ J_{(\lambda)} = \ln N \text{ if } \lambda_{mm'} = \frac{1}{N}. \end{array}$$

Hence, the quantity $J_{(\lambda)}$ vanishes in the case of an ideal measurement of observable $\{M_{m'}\}$, and obtains its maximal value if the measurement is uninformative (i.e. does not yield any information on the initial state) due to maximal disturbance of the measurement results. For a joint nonideal measurement as defined by (9), the nonidealities of both nonideality matrices $(\lambda_{mm'})$ and $(\mu_{nn'})$ can be quantified in a similar way.

For a joint nonideal measurement of two maximal standard observables $A = \sum_m a_m M_m$ and $B = \sum_n b_n N_n$, with M_m and N_n projection operators on eigenvectors $|a_m\rangle$ and $|b_n\rangle$, respectively, the nonideality measures $J_{(\lambda)}$ and $J_{(\mu)}$ obey the following inequality [17]:

$$J_{(\lambda)} + J_{(\mu)} \geq -2 \ln\{max_{mn} |\langle a_m | b_n \rangle|\}. \tag{11}$$

It is evident that (11) is a nontrivial inequality (the right-hand side unequal to zero) if the two observables A and B are incompatible in the sense that the operators do not commute. It is important to note that this inequality is derived from relation (9), and, hence, must be satisfied in any measurement procedure that can be interpreted as a joint nonideal measurement of two incompatible standard observables. The inequality can easily be generalized to nonmaximal standard observables having a discrete spectrum. From the example of "eight-port optical homodyning" it is seen that also in the case of continuous spectra a similar behavior of mutual disturbance can be observed. For this case the problem of finding an inequality like (11) has not been completely solved however [38]. In relation (9) only the *observables* (i.e. the measurement procedures) are involved. Therefore, contrary to the Heisenberg inequality (1), inequality (11) is completely independent of the initial state of the object. Hence, inequality (11) does not refer to the preparation of the initial state, but unequivocally belongs to the inaccuracies of the measurement process.

Inequality (11) should be distinguished from the entropic uncertainty relation [39, 40, 41, 42] for the standard observables $A = \sum_m a_m M_m$ and $B = \sum_n b_n N_n$,

$$H_{\{M_m\}}(\rho) + H_{\{N_n\}}(\rho) \geq -2 \ln\{max_{m,n} |\langle a_m | b_n \rangle|\}, \tag{12}$$

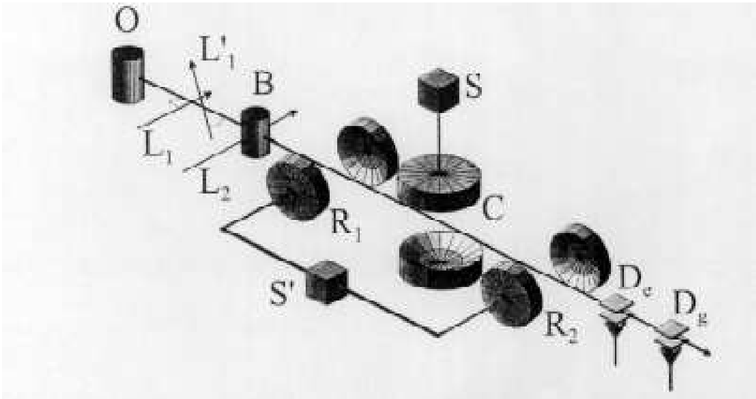

Figure 2: Atomic beam interference experiment of Brune et al.

in which $H_{\{M_m\}}(\rho) = -\sum_m p_m \ln p_m,\ p_m = Tr\rho M_m$ (and analogously for B). Inequality (12), although quite similar to inequality (11), should be compared with the Heisenberg inequality (1), expressing a property of the initial state ρ, to be tested by means of separate measurements of observables $\{M_m\}$ and $\{N_n\}$.

4 ATOMIC BEAM EXPERIMENT AS A JOINT NONIDEAL MEASUREMENT OF INTERFERENCE AND PATH OBSERVABLES

Instead of the classical double-slit experiment we shall consider an interference experiment with atoms, related to the one performed by Brune et al. [43]. Due to the simplicity of its mathematical description this experiment yields a better illustration of the problem of complementarity due to mutual disturbance in a joint measurement of incompatible observables than is provided by the "thought experiment." In the experiment a Rb atom is sent through three cavities, R_1, C and R_2 (cf. figure 2), R_1 and R_2 being (approximately) resonant with a particular transition between two Rydberg states of the atom. Whereas the experiment without cavity C is a pure interference experiment, already performed by Ramsey [44], the introduction of cavity C provides the possibility to obtain also "which-way" information. The visibility of the interference fringes decreases as the field amplitude γ in cavity C increases, but it vanishes only in the limit $\gamma \to \infty$. So, for finite γ information on both interference and path can be obtained. The experiment is analogous to the neutron interference experiments performed by Summhammer et al. [45], in which an absorber is inserted in one of the paths, the absorber playing an analogous role as the cavity C field. The atomic beam experiment differs from the neutron and double-slit experiments to the extent that the paths are not trajectories in configuration space but in the Hilbert space of the internal states of the Rb atom. Mathematically this does not constitute a difference, however. Consequently the present analysis is similar to the analysis of the neutron interference experiments performed by de Muynck and Martens [34].

In the following we restrict to exact resonance of the cavity R_1 and R_2 fields with the transition between the circular Rydberg levels with principal quantum numbers $n = 51$ and $n = 50$, to be indicated as e and g, respectively ($\omega = \omega_e - \omega_g$). We also restrict to an atom velocity for which a $\pi/2$ pulse is generated in these cavities. Then the following transitions are generated in a cavity:

$$|e\rangle \rightarrow \frac{1}{\sqrt{2}}(|e\rangle - i|g\rangle),\ |g\rangle \rightarrow \frac{1}{\sqrt{2}}(-i|e\rangle + |g\rangle).$$

The cavity C field is chosen to be off-resonant, so there is no exchange of energy when the atom passes C. The field in cavity C, initially prepared in a coherent state $|\gamma\rangle$, merely undergoes a phase shift Φ (single atom index effect) which depends on the state of the passing Rb atom in the following way [43]:

$$|e\rangle \otimes |\gamma\rangle \overset{C}{\rightarrow} |e\rangle \otimes \left|\gamma e^{i\Phi}\right\rangle,\ |g\rangle \otimes |\gamma\rangle \overset{C}{\rightarrow} |g\rangle \otimes \left|\gamma e^{-i\Phi}\right\rangle.$$

For the initial state $|\Psi_{in}\rangle = [\alpha\,|e\rangle + \beta\,|g\rangle] \otimes |\gamma\rangle$ of the combined atom-field system we get as the final state after leaving cavity R_2

$$|\Psi_f\rangle = \left\{ \begin{array}{l} |e\rangle\,\frac{1}{2}\left[(\alpha - i\beta)\left|\gamma e^{i\Phi}\right\rangle - (\alpha + i\beta)\left|\gamma e^{-i\Phi}\right\rangle\right] \\ -\,|g\rangle\,\frac{1}{2}\left[(i\alpha + \beta)\left|\gamma e^{i\Phi}\right\rangle + (i\alpha - \beta)\left|\gamma e^{-i\Phi}\right\rangle\right]. \end{array} \right.$$

In the experiment performed by Brune et al. [43] it is measured whether in the final state the atom is in state $|e\rangle$ or $|g\rangle$. For the purpose of interpreting the experiment as a joint nonideal measurement of two incompatible observables it is necessary also to perform a measurement of the final cavity C field (for instance, a measurement of photon number). Let the field measurement be represented by the POVM $\{N_n\}$. Then, by putting

$$\begin{array}{l} p_{en} = \langle\Psi_f\,||e\rangle\,\langle e| \otimes N_n\,|\Psi_f\rangle = \langle\Psi_{in}|\,M_{en}\,|\Psi_{in}\rangle\,, \\ p_{gn} = \langle\Psi_f\,||g\rangle\,\langle g| \otimes N_n\,|\Psi_f\rangle = \langle\Psi_{in}|\,M_{gn}\,|\Psi_{in}\rangle\,, \end{array}$$

the POVM $\{M_{en}, M_{gn}\}$ of the experiment is defined. Defining

$$|v_e\rangle = \left|\gamma e^{i\Phi}\right\rangle - \left|\gamma e^{-i\Phi}\right\rangle, |v_g\rangle = \left|\gamma e^{i\Phi}\right\rangle + \left|\gamma e^{-i\Phi}\right\rangle,$$

we find

$$M_{en} = \frac{1}{4}\left(\begin{array}{cc} \langle v_e|N_n|v_e\rangle & -i\langle v_e|N_n|v_g\rangle \\ i\langle v_g|N_n|v_e\rangle & \langle v_g|N_n|v_g\rangle \end{array}\right), M_{gn} = \frac{1}{4}\left(\begin{array}{cc} \langle v_g|N_n|v_g\rangle & -i\langle v_g|N_n|v_e\rangle \\ i\langle v_e|N_n|v_g\rangle & \langle v_e|N_n|v_e\rangle \end{array}\right).$$

For our purpose it will be convenient to take for the POVM $\{N_n\}$:

$$N_n = \frac{1}{\pi}\int_{\mathrm{C^n}} d^2\alpha\,|\alpha\rangle\,\langle\alpha|\,,$$

$|\alpha\rangle$ coherent states, in which the integrations are taken over regions $\mathrm{C^n}$ of the complex plane. We consider a partition of the complex plane into two regions, corresponding to the upper and lower half-planes, and denoted $n = +$ and $n = -$, respectively. The resulting POVM $\{N_+, N_-\}$ is yielding information on whether the phase transition to $+\Phi$ or to $-\Phi$ has taken place. It can be realized in principle by means of eight-port optical homodyning. In the following we shall restrict to $\Phi = \frac{\pi}{2}$.

Defining $|p_\pm\rangle = \frac{1}{\sqrt{2}}(|e\rangle \pm i|g\rangle)$ the POVM $\{M_{e+}, M_{e-}, M_{g+}, M_{g-}\}$ can be represented according to

$$\begin{array}{l} M_{e+} = \frac{I}{4} + \frac{A}{2}(|p_+\rangle\langle p_+| - |p_-\rangle\langle p_-|) + \frac{C_1}{4}(|e\rangle\langle e| - |g\rangle\langle g|), \\ M_{e-} = \frac{I}{4} - \frac{A}{2}(|p_+\rangle\langle p_+| - |p_-\rangle\langle p_-|) + \frac{C_1}{4}(|e\rangle\langle e| - |g\rangle\langle g|), \\ M_{g+} = \frac{I}{4} + \frac{A}{2}(|p_+\rangle\langle p_+| - |p_-\rangle\langle p_-|) - \frac{C_1}{4}(|e\rangle\langle e| - |g\rangle\langle g|), \\ M_{g-} = \frac{I}{4} - \frac{A}{2}(|p_+\rangle\langle p_+| - |p_-\rangle\langle p_-|) - \frac{C_1}{4}(|e\rangle\langle e| - |g\rangle\langle g|), \end{array}$$

with $A = \frac{1}{2}erf(\gamma)$, and $C_1 = \langle i\gamma| - i\gamma\rangle = e^{-2\gamma^2}$.

We first consider the limits $\gamma = 0$ and $\gamma \to \infty$. For $\gamma = 0$ we have $A = 0$ and $C_1 = 1$. Then the POVM reduces to $\left\{\frac{1}{2}|e\rangle\langle e|, \frac{1}{2}|e\rangle\langle e|, \frac{1}{2}|g\rangle\langle g|, \frac{1}{2}|g\rangle\langle g|\right\}$. This is a trivial refinement of the projection-valued measure $\{|e\rangle\langle e|, |g\rangle\langle g|\}$, measured in the Ramsey experiment [44]. We shall refer to the latter observable as the *interference* observable. In the limit $\gamma \to \infty$ we get $A = \frac{1}{2},\ C_1 = 0$. The POVM once more reduces to a trivial refinement of a PVM. We get $\{\frac{1}{2}|p_+\rangle\langle p_+|, \frac{1}{2}|p_+\rangle\langle p_+|, \frac{1}{2}|p_-\rangle\langle p_-|, \frac{1}{2}|p_-\rangle\langle p_-|\}$. Because of the analogy with neutron interference experiments [34] the PVM $\{|p_+\rangle\langle p_+|, |p_-\rangle\langle p_-|\}$ is referred to as the *path* observable.

By ordering the POVM $\{M_{e+}, M_{e-}, M_{g+}, M_{g-}\}$ into a bivariate form according to

$$\{M_{mn}\} := \begin{pmatrix} M_{e+} & M_{g+} \\ M_{e-} & M_{g-} \end{pmatrix} \tag{13}$$

the measurement can be interpreted as a joint nonideal measurement of the interference and path observables for arbitrary γ. Thus, calculating the marginals of the bivariate POVM (13) we find

$$\begin{pmatrix} M_{e+} + M_{e-} \\ M_{g+} + M_{g-} \end{pmatrix} = \overbrace{\frac{1}{2}\begin{pmatrix} 1 + C_1 & 1 - C_1 \\ 1 - C_1 & 1 + C_1 \end{pmatrix}}^{(\lambda_{mn})} \begin{pmatrix} |e\rangle\langle e| \\ |g\rangle\langle g| \end{pmatrix} \tag{14}$$

and

$$\begin{pmatrix} M_{e+} + M_{g+} \\ M_{e-} + M_{g-} \end{pmatrix} = \overbrace{\frac{1}{2}\begin{pmatrix} 1 + 2A & 1 - 2A \\ 1 - 2A & 1 + 2A \end{pmatrix}}^{(\mu_{mn})} \begin{pmatrix} |p_+\rangle\langle p_+| \\ |p_-\rangle\langle p_-| \end{pmatrix}. \tag{15}$$

Hence, the marginals satisfy the conditions (9) of a joint nonideal measurement of the (incompatible) interference and path observables, with nonideality matrices (λ_{mn}) and (μ_{mn}).

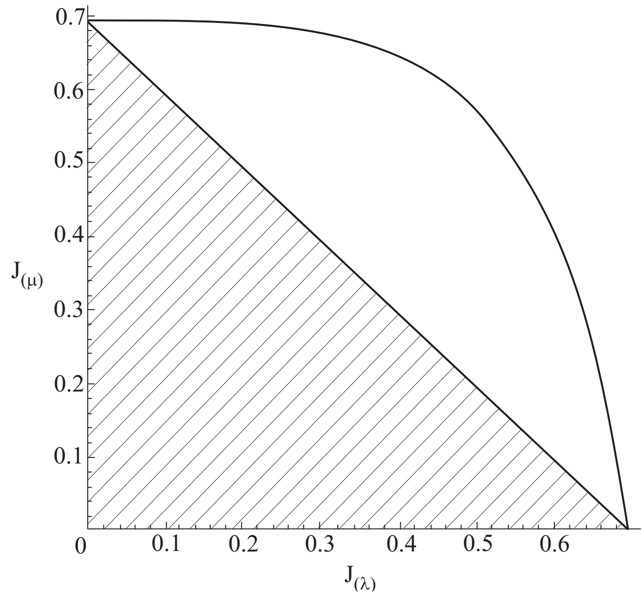

Figure 3: Parametric plot of $J_{(\lambda)}$ versus $J_{(\mu)}$, for $0 \leq \gamma < \infty$. The shaded area is the region that is forbidden by inequality (11).

For the nonideality measures $J_{(\lambda)}$ and $J_{(\mu)}$ defined by (10) we obtain for the nonideality matrices of (14) and (15), respectively,

$$\begin{array}{l} J_{(\lambda)} = -\frac{(1+C_1)}{2}\ln(\frac{1+C_1}{2}) - \frac{(1-C_1)}{2}\ln(\frac{1-C_1}{2}), \\ J_{(\mu)} = -\frac{(1+2A)}{2}\ln(\frac{1+2A}{2}) - \frac{(1-2A)}{2}\ln(\frac{1-2A}{2}). \end{array}$$

From the parametric plot in figure 3 it can be seen that inequality (11) is satisfied. This illustrates the impossibility that both nonideality measures $J_{(\lambda)}$ and $J_{(\mu)}$ jointly have a small value. Figure 3 clearly illustrates the idea of complementarity as this arises in the "thought experiments." If γ is varied, then the measurement arrangement is altered. For Bohr this would signify a different definition of the path and interference observables for each different value of γ, the "latitudes" of the definition of the observables depending on γ. For Heisenberg the interference observable is disturbed more by the measurement process as γ increases (and, hence, C_1 is decreasing), whereas the path observable is disturbed less because A is increasing. Both would interpret this as an expression of the complementarity of the interference and path observables, due to the fact that the operators $|e\rangle\langle e|$ and $|g\rangle\langle g|$ do not commute with $|p_+\rangle\langle p_+|$ and $|p_-\rangle\langle p_-|$. Evidently, the γ dependence of the nonideality matrices $(\lambda_{mm'})$ and $(\mu_{nn'})$ precisely expresses the complementarity that is connected with the mutual disturbance in a joint nonideal measurement of the incompatible interference and path observables.

It is also clearly seen that there is a difference with Heisenberg's disturbance ideas. In the atomic beam interference experiment the nonidealities do not refer to the final object state, but to the information obtained on the *initial* state. Hence, these quantities do not refer to the preparative aspect of measurement (as is the case in Heisenberg's interpretation of his uncertainty relation), but to the determinative one. Contrary to the standard formalism the generalized formalism as embodied by (9) is capable of referring to the past (rather than to the future), even if measurements are involved in which measurement disturbance (inaccuracy) plays an important role. The generalized formalism enables us to consider quantum mechanical measurements in the usual determinative sense, and allows us to distinguish this determinative aspect from the question in which (postmeasurement) state the object is prepared by the measurement.

5 DISCUSSION AND CONCLUSIONS

For unbiased nonideal measurements, i.e. measurements for which the nonideal and the ideal versions in (5) yield the same expectation values for operators $\sum_m a_m M_m$ and $\sum_m a_m R_m$, the nonideality matrix $(\lambda_{mm'})$ should satisfy the equality $a_{m'} = \sum_m a_m \lambda_{mm'}$. If we restrict to unbiased nonideal measurements it also is possible by using standard deviations rather than the entropic quantities (10) to demonstrate that there are two sources of uncertainty. Thus, using the notation $r_m = Tr\rho R_m$, $p_m = Tr\rho M_m$, the relation $r_m = \sum_{m'} \lambda_{mm'} p_{m'}$ between the probability distributions $\{p_m\}$ (of the ideal measurement) and $\{r_m\}$ (obtained in the nonideal one) is found from (5). For unbiased measurements the standard deviation of the measurement results a_m of observable $A = \sum_m a_m M_m$, obtained in the nonideal measurement, can easily be seen to satisfy the equality

$$\Delta(\{r_m\})^2 = \Delta(\{p_m\})^2 + \sum_{m'} \Delta^2_{m'} p_{m'}, \tag{16}$$

with

$$\Delta^2_{m'} := \sum_m a_m^2 \lambda_{mm'} - (\sum_m a_m \lambda_{mm'})^2.$$

The quantity (16) consists of two different contributions: (i) the contribution $\Delta(\{p_m\})^2 = (\Delta A)^2$ obtained in an ideal measurement, which is independent of the parameters of the measurement arrangement, and, for this reason interpretable in the usual way as a property of the initial state of the object, and (ii) a contribution $\sum_{m'} \Delta^2_{m'} p_{m'}$ due to the nonideality of the measurement procedure. Also, it is not difficult to see that

$$\Delta(\{r_m\}) \geq \Delta(\{p_m\}).$$

If in a joint nonideal measurement of two incompatible observables A and B both nonideal measurements are unbiased, then for the joint nonideal measurement the *generalized* Heisenberg uncertainty relation

$$\Delta(\{r_m\})\Delta(\{s_n\}) \geq \Delta A \Delta B \geq \frac{1}{2} \mid \langle [A, B]_- \rangle \mid \tag{17}$$

($\{s_n\}$ the nonideally measured probability distribution of observable B) immediately follows from the Heisenberg relation (1).

A disadvantage of (17) is that not all nonideal measurements are unbiased. For instance, as easily follows from (8), detector inefficiency will cause the average measured photon number to be smaller than the ideal one. For this reason (17) is not universally applicable. Another disadvantage is that in the expressions for $\Delta(\{r_m\})$ and $\Delta(\{s_n\})$ the two contributions to uncertainty are merged into one single quantity. An inequality that is valid for biased measurements too might be obtained by combining the entropic uncertainty relation (12) with inequality (11), thus yielding an equality,

$$(H_{\{M_m\}} + J_{(\lambda)}) + (H_{\{N_n\}} + J_{(\mu)}) \geq -4 \ln\{max_{mn}|\langle a_m|b_n\rangle|\}, \tag{18}$$

that can be compared to the generalized Heisenberg inequality (17). However, it is evident that it is not very meaningful to do this because in (18) the two different contributions are once again merged, thus veiling their different origins. As follows from (12) and (11) both sources satisfy their own inequality. The opportunity entropic quantities offer for exhibiting this seems to be an important advantage of these quantities over the widely used standard deviations. It has occasionally been noted ([16], p. 153, and [26]) that for specific measurement procedures an uncertainty relation for the joint measurement of incompatible observables can be formulated in terms of standard deviations. It is not at all clear, however, whether such a relation exists, which is comparable to inequality (11), and valid for *all* quantum mechanical measurements interpretable as joint measurements of incompatible standard observables.

Failure to distinguish the different contributions to uncertainty represented by the different terms in (16) and (18) is at the basis of the Copenhagen confusion with respect to the uncertainty relations originating with the discussion of the double-slit experiment. Because no clear distinction was drawn between preparation and measurement, these could not be properly distinguished as different sources of "uncertainty", *both* contributing in their own way. Since inequality (3) refers to the measurement process rather than to the preparation of the initial state, it should be compared to inequality (11) rather than to the Heisenberg one. The fact that (3) has the same mathematical form as (4) is caused by the more or less accidental circumstance that in the double-slit example the uncertainty induced by the measurement process is a consequence of the *preparation* uncertainty of a part of the measurement arrangement (viz. screen S) described by (2). However, as demonstrated by the atomic beam interference example, the measurement disturbance seems to originate more generally from the quantum mechanical character of the whole interaction process of object and measuring instrument. Fluctuations of the latter may be a part of this, but need not always play an essential role in the complementarity issue.

It is important to stress that inequality (11) is obtained from the *generalized* formalism, being capable of describing measurements represented by POVMs. The founders of the Copenhagen interpretation did not dispose of this formalism. Indeed, in the "thought experiments" a measurement is always thought to be represented by a selfadjoint operator (i.e. a projection-valued measure). In the example of the atomic beam interference experiment this implies a restriction to the extreme values $\gamma = 0$ and $\gamma \rightarrow \infty$. The restriction to these extreme values was responsible for the view in which interference is completely disturbed in a "which-way" measurement (and vice versa). This, indeed, is confirmed by the limiting values of the nonideality matrices given in (14) and (15), yielding an uninformative marginal for path if interference is measured ideally (and vice versa). In the intermediate region $0 < \gamma < \infty$ information on both observables is obtained, be it that this information is disturbed in the way described by inequality (11).

From the generalized formalism it is clear that in the atomic beam interference experiment complementarity of the interference and path observables is at stake. Nevertheless, as is evident from the recent discussion referred to above [10, 11, 12, 46], this effect is still sometimes associated with the Heisenberg inequality for position and momentum. It seems that in this discussion the confusion between complementarity of preparation and measurement still exists. Of course, since a measurement may also be a preparation procedure for a postmeasurement state of the object, the Heisenberg inequality $\Delta Q \Delta P \geq \hbar/2$ (as well as inequality (1) for any choice of observables A and B) should also hold in the postmeasurement object state. This, however, is independent of this procedure being a *measurement*. As a matter of fact, Q and P must satisfy the Heisenberg inequality in the postmeasurement state independently of which observables A and B have been measured jointly.

Complementarity in the sense of mutual disturbance in a joint measurement of incompatible observables, as characterized by inequality (11), does not refer to the preparation of the postmeasurement state, but to a restriction with respect to obtaining information on the *initial* object state. Apart from this difference, inequality (11) nevertheless seems to be a mathematical expression of the Copenhagen concept of complementarity, viz. mutual disturbance in a joint (or simultaneous) *measurement* of incompatible observables. It seems that the physical intuition that was expressed by the "thought experiments" was perfect in this respect. However, confusion had to arise because of the impossibility of dealing with joint measurements of incompatible observables using the standard formalism. Bohr and Heisenberg were led astray by the availability of the uncertainty relation (4) (or, more generally, (1)) following from the latter formalism, unjustifiedly thinking that this relation provided a materialization of their physical intuition.

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FIGURE CAPTIONS

Figure 1: Double-slit experiment.

Figure 2: Atomic beam interference experiment of Brune et al.

Figure 3: Parametric plot of $J_{(\lambda)}$ versus $J_{(\mu)}$, for $0 \leq \gamma < \infty$. The shaded area is the region that is forbidden by inequality (11).