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Neutron interferometry and the joint measurement of incompatible observables

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Different neutron-interferometric setups for the joint detection of "path" and "interference" are investigated from the point of view of the theory of joint measurement. It is seen that this joint detection can only be done with limited quality. This substantiates Bohr's insistence that the impossibility of jointly seeing interference and path is a consequence of the uncertainty principle. A connection between joint measurement and Wigner distributions is noted.

INTRODUCTION

Recently neutron-interferometric experiments[1–5] have been extended to the experimental domain in which both particle and wave properties of the neutron are studied in the same experiment.[3–5] This is done by inserting an absorber into one of the two interferometer beams, with the intention of obtaining a (partial) determination of the neutron path without completely wiping out the interference pattern. In neutron interference experiments a neutron beam is split in an interferometer (cf. Fig. 1) into two beams, one of which can be phase shifted so as to realize phase-dependent detection probabilities in the neutron detectors D_A and D_B, to be indicated as the interference pattern (see, for instance, Fig. 2 of Ref. 4). An aluminum plate can be used as a phase shifter, the phase being varied by varying the path length of the neutron path in the plate. This is done by rotating the plate. In the experiments described in Ref. 4, a single-crystal silicon interferometer is used by which a beam separation of a few centimeters can be achieved. Due to this large separation, it is possible to physically interact with the neutrons in either the left- or the right-hand beam separately, and study the influence of this interaction on the interference.

In particular, it is an interesting question what happens with the interference pattern if it is determined by which path the neutron traversed the interferometer. It is often said that by such a determination the interference pattern is wiped out completely. This indeed is the case if one of the paths is blocked completely, so that we know with certainty that the neutron went the other way. However, as predicted theoretically by Wootters and Zurek,[6] and demonstrated experimentally by Mittelstaedt, Prieur, and Schieder[7] in the case of photon interference, if the path is determined with a probability less than 1, the visibility of the interference pattern may be preserved to a certain extent.

The experiments of Summhammer, Rauch, and Tuppinger[4] are essentially of the latter kind. In these experiments an absorbing element is inserted into one of the beams (cf. Fig. 2), so as to decrease the probability that a

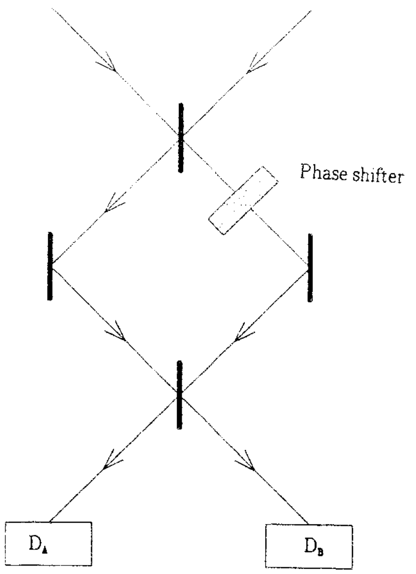

FIG. 1. Basic interferometer setup. A neutron beam is sent along one of the incoming paths, and then coherently split. One of the partial beams undergoes a phase shift χ. The beams are then coherently mixed.

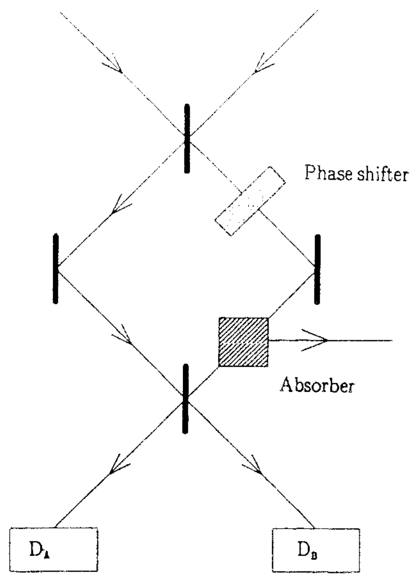

FIG. 2. Extended setup. Neutrons are deleted (either stochastically or deterministically) with probability $1-a$ or $1-v$ from the right partial beam.

neutron reaching the detectors did so via the path containing the absorber. In this paper we will discuss two different kinds of absorption studied in Ref. 4, viz. stochastic and deterministic absorption. In the stochastic absorption experiment the absorber consists of a stack of 1-mm-thick slabs of gold or indium permitting the transmission probability to be varied between 0.9% and 48.0% by stacking different numbers of slabs. In the deterministic case the absorber is a chopper consisting of a rotating disk of 1-mm-thick cadmium, a number of octant sections having been cut out. Since the closed position of the chopper ensures an absorption probability of better than 99.99%, the neutron is either transmitted or stopped depending on whether it finds the chopper in an open or a closed position. In Fig. 2 of Ref. 4 the experimental interference patterns for stochastic and deterministic absorption are compared for approximately equal transmission probabilities.

NONIDEAL MEASUREMENTS

Realistic measurements can best be described by means of so-called positive-operator-valued measures (POVM's).[8] POVM's are generalizations of the projection-valued measures (PVM's) which are usually considered to represent quantum-mechanical observables (i.e., the orthogonal spectral representations of self-adjoint operators). A POVM is a set of operators $\{\underline{M}_k\}$ satisfying

$$\underline{M}_k \geq 0, \quad \sum_k \underline{M}_k = \underline{1} , \tag{1}$$

in which $\underline{M}_k$ is not necessarily a projection operator, and in general $[\underline{M}_k, \underline{M}_j]_- \neq \underline{0}$, $k \neq j$. The detection probabilities of the experiment are given by $p_k = \mathrm{Tr}(\underline{\rho M}_k)$.

A POVM $\{\underline{M}_k\}$ is said to represent a nonideal measurement[9] of an observable described by the POVM $\{\underline{N}_l\}$ if a stochastic matrix (λ_{kl}) exists, such that

$$\underline{M}_k = \sum_l \lambda_{kl} \underline{N}_l, \quad \lambda_{kl} \geq 0, \quad \sum_k \lambda_{kl} = 1 . \tag{2}$$

The two POVM's will be called equivalent if, apart from (2), we also have

$$\underline{N}_l = \sum_k \mu_{lk} \underline{M}_k, \quad \mu_{lk} \geq 0, \quad \sum_l \mu_{lk} = 1 . \tag{2'}$$

Evidently, equivalent POVM's describe measurements yielding the same statistical information about the object, with equal quality.[9] For example, two POVM's differing only in labeling are equivalent. Another example of equivalence is given through the notion of *equivalent refinement*. A bivariate POVM $\{\underline{M}_{ik}\}$ is called an equivalent refinement of the POVM $\{\underline{M}_k\}$ if $\underline{M}_{ik} = \alpha_{ik} \underline{M}_k$, $0 \leq \alpha_{ik} \leq 1$, and $\sum_i \alpha_{ik} = 1$.

If $\{\underline{M}_k\}$ and $\{\underline{N}_l\}$ are not equivalent, it may nevertheless be possible to invert the relation (2) according to $\underline{N}_l = \sum_k (\lambda^{-1})_{lk} \underline{M}_k$, in which $[(\lambda^{-1})_{lk}]$ is the inverse matrix of (λ_{kl}). In general, the matrix $[(\lambda^{-1})_{lk}]$ is not a stochastic matrix, its matrix elements not even being positive. So, if (2′) is not satisfied together with (2), $\{\underline{N}_l\}$ does not describe a nonideal measurement of $\{\underline{M}_k\}$. If the relation (2) does, however, have an inverse, we call the measurement $\{\underline{M}_k\}$ an invertibly nonideal measurement of $\{\underline{N}_l\}$, because of the possibility of obtaining exact information on $\{\underline{N}_l\}$ by measuring $\{\underline{M}_k\}$ and calculating all the $\{\underline{N}_l\}$ probabilities by means of the inverse matrix $[(\lambda^{-1})_{lk}]$.[9]

A bivariate POVM $\{\underline{R}_{mn}\}$ represents a *joint nonideal measurement*[10] of two POVM's $\{\underline{M}_k\}$ and $\{\underline{N}_l\}$, if its marginals $\{\sum_n \underline{R}_{mn}\}$ and $\{\sum_m \underline{R}_{mn}\}$ represent nonideal measurements of $\{\underline{M}_k\}$ and $\{\underline{N}_l\}$, respectively, i.e., if two stochastic matrices (λ_{mk}) and (μ_{nl}) exist, such that

$$\begin{aligned} \sum_n \underline{R}_{mn} &= \sum_k \lambda_{mk} \underline{M}_k, \quad \lambda_{mk} \geq 0, \quad \sum_m \lambda_{mk} = 1 , \\ \sum_m \underline{R}_{mn} &= \sum_l \mu_{nl} \underline{N}_l, \quad \mu_{nl} \geq 0, \quad \sum_n \mu_{nl} = 1 . \end{aligned} \tag{3}$$

In Ref. 10 an inequality was derived that can unambiguously be interpreted as limiting the inaccuracies achievable in a joint measurement of two PVM's on a finite-dimensional Hilbert space. These inaccuracies are embodied in a deviation of the stochastic matrices (λ_{mk}) and (μ_{nl}) in (3) from the unit matrix. It turns out that these matrices satisfy a principle of complementarity with respect to this deviation: they cannot both approach the unit matrix if $\{\underline{M}_k\}$ and $\{\underline{N}_l\}$ are incompatible.

We shall demonstrate that the above-mentioned neutron-interference experiments fit into this scheme. We show that these experiments also exhibit complementary behavior. In order to do this in a quantitative way we must introduce some measure representing the inaccuracy embodied in the matrices of relation (3). An example is the mean row entropy

$$\delta_{(\lambda)} = -\frac{1}{n} \sum_{m,k} \lambda_{mk} \ln \left[\frac{\lambda_{mk}}{\sum_j \lambda_{mj}} \right] , \tag{4a}$$

where n is the number of elements in $\{\underline{M}_k\}$. For square nonideality matrices we can also take as such a measure the quantity

$$\epsilon_{(\lambda)} = 1 - \prod |\nu(\lambda_{mk})| . \tag{4b}$$

where $\nu(\lambda_{mk})$ represents the eigenvalues of (λ_{mk}). The measure $\epsilon_{(\lambda)}[\delta_{(\lambda)}]$ takes values between 0 and 1 [between 0 and $\ln(n)$]. For λ_{mk} equal to the unit matrix we obtain $\epsilon_{(\lambda)}=0$ $[\delta_{(\lambda)}=0]$, whereas $\epsilon_{(\lambda)}=1$ $[\delta_{(\lambda)}=\ln(n)]$ if $\lambda_{mk} = \tilde{\lambda}_m$, i.e., for maximal inaccuracy. The inaccuracy relation presented in Ref. 10 used $\delta_{(\lambda)}$. An inaccuracy relation using $\epsilon_{(\lambda)}$ has as yet not been derived.

Unfortunately both versions of the absorption experiments mentioned before do not yield information beyond that given by the separate observations (in this case we call a joint measurement a trivial one). The deterministic absorption experiment can even be seen as an experiment in which *either* path *or* interference is measured in two subensembles, instead of a true *joint* measurement. Therefore we consider a third setup, containing a quantum nondemolition (QND) detection of the neutron's path.[2,11] This also fits into our formalism, but does not have the above-mentioned drawback.

NEUTRON INTERFERENCE WITHOUT ABSORPTION

We first consider the basic neutron interference experiment without absorption.[3] A neutron beam enters on one of the indicated paths (Fig. 1), and is passed coherently through two beamsplitters. In between these the phase of one of the partial beams is shifted by an amount χ. The two partial beams are allowed to interfere in a third beamsplitter, after which the neutrons are registered by one of the detectors D_A and D_B (cf. Fig. 1). Ignoring the neutron's polarization, the experiment can be completely described by the two-dimensional Hilbert space spanned by the orthogonal state functions $|L\rangle$ and $|R\rangle$ representing the left and right beam, respectively. Assuming[5] that each reflection introduces a phase shift $\frac{1}{2}\pi$, the interferometer's action can be described by

$$\begin{aligned}|L\rangle &\to \tfrac{1}{2}\sqrt{2}(i|L\rangle+|R\rangle)\\ &\to \tfrac{1}{2}\sqrt{2}[i|L\rangle+\exp(i\chi)|R\rangle]\\ &\to \tfrac{1}{2}i\{[-1+\exp(i\chi)]|L\rangle\\ &\qquad +[i+i\exp(i\chi)]|R\rangle\}\ . \end{aligned} \tag{5}$$

Thus the final probabilities for finding the beam in the detectors are $\frac{1}{2}\mp\frac{1}{2}\cos(\chi)$.

More generally, if the incoming state is $|\psi_{\text{in}}\rangle=\alpha|L\rangle+\beta|R\rangle$, the outgoing state is given by

$$\begin{aligned}|\psi_{\text{out}}\rangle &= \tfrac{1}{2}i\{\alpha[-1+\exp(i\chi)]+\beta[i+i\exp(i\chi)]\}|L\rangle\\ &\quad +\tfrac{1}{2}i\{\alpha[i+i\exp(i\chi)]+\beta[1-\exp(i\chi)]\}|R\rangle\ .\end{aligned} \tag{6}$$

The detection probabilities of detectors D_A and D_B are then given by

$$\begin{aligned}p_A &= \mathrm{Tr}(\underline{\rho}_{\text{in}}\underline{P}_A)=|\langle L|\psi_{\text{out}}\rangle|^2\ ,\\ p_B &= \mathrm{Tr}(\underline{\rho}_{\text{in}}\underline{P}_B)=|\langle R|\psi_{\text{out}}\rangle|^2\ ,\end{aligned} \tag{7}$$

and

$$\underline{P}_A=\begin{pmatrix}\sin^2(\frac{1}{2}\chi) & \frac{1}{2}\sin(\chi)\\ \frac{1}{2}\sin(\chi) & \cos^2(\frac{1}{2}\chi)\end{pmatrix}=\underline{1}-\underline{P}_B\ . \tag{8}$$

The observable represented by $\{\underline{P}_A,\underline{P}_B\}$ can be interpreted as the quantum-mechanical observable measured in an interference experiment. Since $\underline{P}_A$ and $\underline{P}_B$ are orthogonal projections, $\{\underline{P}_A,\underline{P}_B\}$ is a PVM.

NEUTRON INTERFERENCE WITH STOCHASTIC ABSORPTION

In the case of stochastic absorption a partial absorber is inserted in one of the beams (Fig.2). This induces the possibility that the neutron does not enter either detector D_A or D_B, but is transferred to a third mode, described by $|Z\rangle$, orthogonal to both $|L\rangle$ and $|R\rangle$. Assuming that the absorber does not induce an extra phase shift (if a phase shift occurs, then it can be accommodated by a constant additional contribution to χ), if a is the transmission probability, we obtain for the outgoing state

$$\begin{aligned}|\psi_{\text{out}}\rangle &= \tfrac{1}{2}i\{\alpha[-1+\sqrt{a}\exp(i\chi)]+\beta[i+i\sqrt{a}\exp(i\chi)]\}|L\rangle\\ &\quad +\tfrac{1}{2}i\{\alpha[i+i\sqrt{a}\exp(i\chi)]+\beta[1-\sqrt{a}\exp(i\chi)]\}|R\rangle+\tfrac{1}{2}i\sqrt{2(1-a)}(\alpha+i\beta)|Z\rangle\ ,\end{aligned} \tag{9}$$

yielding

$$p_A=\mathrm{Tr}(\underline{\rho}_{\text{in}}\underline{M}_A)=|\langle L|\psi_{\text{out}}\rangle|^2,\ \ p_B=\mathrm{Tr}(\underline{\rho}_{\text{in}}\underline{M}_B)=|\langle R|\psi_{\text{out}}\rangle|^2,\ \ p_Z=\mathrm{Tr}(\underline{\rho}\,\underline{M}_Z)=|\langle Z|\psi_{\text{out}}\rangle|^2\ , \tag{10}$$

and

$$\begin{aligned}\underline{M}_A &= \tfrac{1}{4}\begin{pmatrix}(1+a)-2\sqrt{a}\cos(\chi) & -i(1-a)+2\sqrt{a}\sin(\chi)\\ i(1-a)+2\sqrt{a}\sin(\chi) & 1+a+2\sqrt{a}\cos(\chi)\end{pmatrix},\\ \underline{M}_B &= \tfrac{1}{4}\begin{pmatrix}(1+a)+2\sqrt{a}\cos(\chi) & -i(1-a)-2\sqrt{a}\sin(\chi)\\ (1-a)-2\sqrt{a}\sin(\chi) & (1+a)-2\sqrt{a}\cos(\chi)\end{pmatrix},\\ \underline{M}_Z &= \tfrac{1}{2}(1-a)\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}.\end{aligned} \tag{11}$$

For $a=1$ this POVM reduces to the "interference" observable $\{\underline{P}_A,\underline{P}_B,\underline{0}\}$. For $a=0$ we obtain $\{\frac{1}{2}\underline{P}_+,\frac{1}{2}\underline{P}_+,\underline{P}_-\}$, in which $\underline{P}_+=1-\underline{P}_-$ is the projection operator $\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$, corresponding to the neutron passing on the left-hand side after the first beamsplitter. So, an equivalent refinement of the "path" observable $\{\underline{P}_+,\underline{P}_-\}$ is obtained when $a=0$. For $\alpha=0$, $\beta=1$ we obtain from (10) and (11)

$$p_A=\tfrac{1}{4}(1+a+2\sqrt{a}\,\cos\chi)\ , \tag{12}$$

in accordance with formula (8) of Ref. 4.

In order to be able to interpret the neutron interference experiment as a joint nonideal measurement of the interference and path observables, we reorder the POVM $\{\underline{M}_A,\underline{M}_B,\underline{M}_Z\}$ into the equivalent refinement

$$\begin{pmatrix}\underline{M}_A & \underline{M}_B\\ \frac{1}{2}\underline{M}_Z & \frac{1}{2}\underline{M}_Z\end{pmatrix}\ . \tag{13}$$

This is a bivariate POVM, the marginals of which can be calculated to yield

$$\begin{pmatrix}\underline{M}_A+\underline{M}_B\\ \underline{M}_Z\end{pmatrix}=(\lambda_{mk})\begin{pmatrix}\underline{P}_+\\ \underline{P}_-\end{pmatrix}\ ,$$

$$(\lambda_{mk})=\begin{pmatrix}1 & a\\ 0 & 1-a\end{pmatrix}\ , \tag{14}$$

and

$$\begin{pmatrix}\underline{M}_A+\frac{1}{2}\underline{M}_Z\\ \underline{M}_B+\frac{1}{2}\underline{M}_Z\end{pmatrix}=(\mu_{nl})\begin{pmatrix}\underline{P}_A\\ \underline{P}_B\end{pmatrix}\ ,$$

$$(\mu_{nl})=\frac{1}{2}\begin{pmatrix}1+\sqrt{a} & 1-\sqrt{a}\\ 1-\sqrt{a} & 1+\sqrt{a}\end{pmatrix}\ . \tag{15}$$

This precisely fits our definition (3) of a joint nonideal measurement of interference and path observables. It is seen that in the case $a=0$ the marginal (14) reduces to the path observable $\{\underline{P}_+,\underline{P}_-\}$, whereas (15) yields the interference observable $\{\underline{P}_A,\underline{P}_B\}$ if $a=1$.

Now we can evaluate the inaccuracy measure (4b) for these two matrices, obtaining

$$\epsilon_{(\lambda)}=a,\quad \epsilon_{(\mu)}=1-\sqrt{a}\ . \tag{16}$$

As a function of the parameter a, this tracks out a curve in $\mathbb{R}^2$, which indicates the complementarity in the experiment because of the impossibility that both $\epsilon_{(\lambda)}$ and $\epsilon_{(\mu)}$ approach the value zero [Fig. 3(b)]. For the measure (4a) the analogous curve is shown in Fig. 3(a) (solid line). In this figure we also indicated the quantum limit of the inaccuracy relation of Ref. 10: according to this relation it is impossible to find *any* joint nonideal measurement procedure of interference and path with the pair $(\delta_{(\mu)},\delta_{(\lambda)})$ of the inaccuracy measures (4a) in the shaded region of Fig. 3(a). It must be emphasized here that this result, representing the complementarity of these two observables, is derived from a relation that is different from the Heisenberg uncertainty relation. As a matter of fact, the inaccuracy measure (4a) is a property of the observable only. Contrary to the standard deviation, it is independent of the state function and, for this reason, seems to be more suitable to represent a notion of measurement accuracy that is independent of the uncertainty induced by the preparation.

Outside the limits $a=1$ and 0 both stochastic matrices (λ_{mk}) and (μ_{nl}), in (14) and (15), have inverses. This implies that the nonideal measurements described by (14) and (15) are invertible. Consequently, it is possible to calculate the probabilities of the ideal interference and path observables from the data obtained in the measurement considered here. So, notwithstanding the complementarity discussed before, the experiment can be interpreted in this sense as a joint measurement of interference and path.

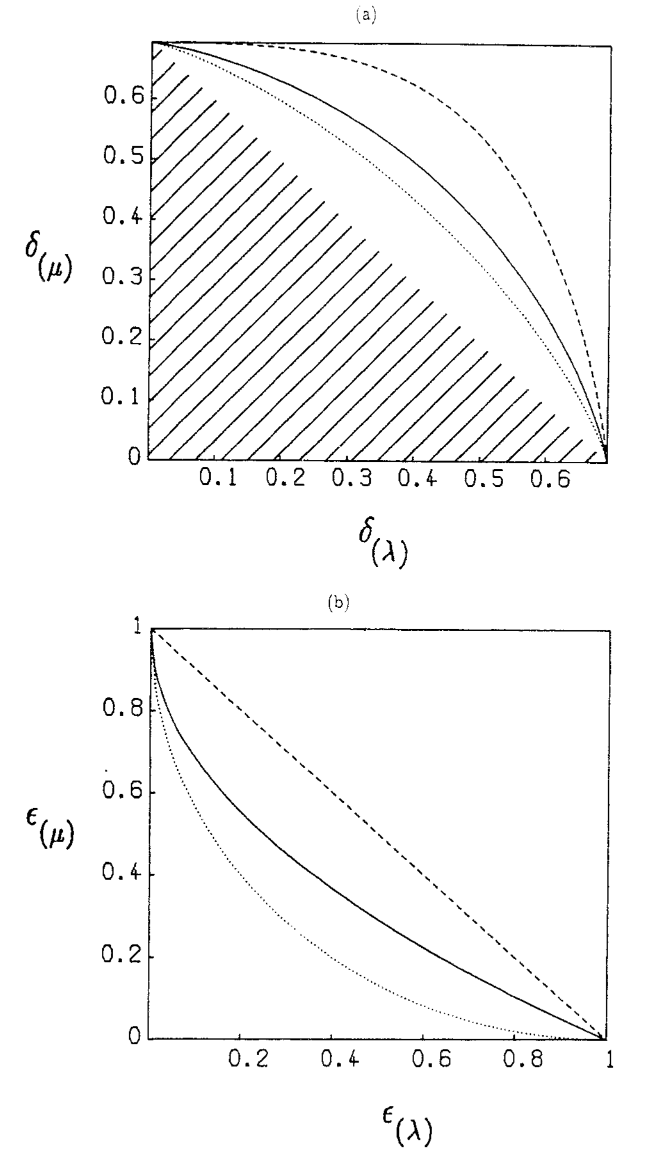

FIG. 3. Complementarity of path and interference using inaccuracy measures (4). In (a) the shaded area represents the $\delta_{(\lambda)}-\delta_{(\mu)}$ combinations that are prohibited by the uncertainty principle (Ref. 7). (b) For $\epsilon_{(\lambda)}-\epsilon_{(\mu)}$ such an inaccuracy relation has not yet been derived. The curves indicate the performance of deterministic (dashed lines) and stochastic (solid lines) absorption setups, and of the QND setup (dotted lines).

NEUTRON INTERFERENCE WITH DETERMINISTIC ABSORPTION

In a different neutron-interference experiment the stochastic absorber is replaced by a beam chopper.[3,4] By this chopper the particle is either left completely undisturbed (probability v) or is completely absorbed. This can be interpreted as a measurement of the observable described by the bivariate POVM

$$\{\underline{M}'_i\} = [v\underline{M}_i(a=1) + (1-v)\underline{M}_i(a=0)], \quad i = A,B,Z \tag{17}$$

with $\{\underline{M}_i\}$ being given by (11). For $i = A$ this yields

$$\underline{M}'_A = \frac{1}{4}\begin{bmatrix} 1+v-2v\cos(\chi) & -i(1-v)+2v\sin(\chi) \\ i(1-v)+2v\sin(\chi) & 1+v+2v\cos(\chi) \end{bmatrix}, \tag{18}$$

which has to be compared with the first matrix of (11). From (18) the detection probability of detector D_A for $\alpha=0, \beta=1$ is calculated, analogous to (12), as

$$p'_A = \frac{1-v}{4} + \tfrac{1}{2}v[1+\cos(\chi)], \tag{19}$$

which is in accord with formula (9) of Ref. 4. Written in this way we see that p'_A consists of a contribution $[\sim(1-v)]$ stemming from "labeled" neutrons[4] that do not contribute to the interference because their path is known accurately, and a contribution $(\sim v)$ having a maximal interference amplitude as in the pure interference experiment described by (8).

In Ref. 4 the peculiar difference was noticed between (12) and (19) as a function of the transmission probabilities a and v, respectively. The results illustrated by Figs. 3(a) and 3(b) corroborate the conclusion of Ref. 4 that the interference is smaller in the case of deterministic absorption. This difference highlights the difference of the two measurement processes, deterministic absorption just corresponding to a classical mixture of the two extremes $a=1$ and 0, whereas in stochastic absorption the absorption process is fundamentally quantum mechanical, hence wiping out phase relations less effectively. In this connection we may refer to the paper by Mittelstaedt, Prieur, and Schieder[7] in which it is demonstrated in an analogous interference experiment that the information on one observable is not always destroyed completely by the measurement of an incompatible observable. The neutron-interference experiment with stochastic absorption evidently has comparable characteristics.

Analogously to (14) and (15) the POVM (17) turns out to represent a joint nonideal measurement of the interference and path observables, the nonideality matrices being found as convex combinations $\lambda'_{mk} = v\lambda_{mk}(a=1) + (1-v)\lambda_{mk}(a=0)$ and an analogous expression for μ'_{nl}, yielding

$$(\lambda'_{mk}) = \begin{bmatrix} 1 & v \\ 0 & 1-v \end{bmatrix} \tag{20}$$

and

$$(\mu'_{nl}) = \begin{bmatrix} \frac{(1+v)}{2} & \frac{(1-v)}{2} \\ \frac{(1-v)}{2} & \frac{(1+v)}{2} \end{bmatrix}. \tag{21}$$

Then the amounts (4b) of nonideality are

$$\epsilon_{(\lambda)} = v, \quad \epsilon_{(\mu)} = 1-v . \tag{22}$$

Comparing (22) with (16), we indeed see that this experiment is slightly worse than the stochastic absorption experiment in the sense that it is more distant from the point $\epsilon_{(\lambda)} = \epsilon_{(\mu)} = 0$ than the latter experiment [cf. Fig. 3(b)]. This is also expressed in Fig. 3(a) by a $\delta_{(\lambda)} - \delta_{(\mu)}$ curve.

(NON)TRIVIALITY OF JOINT MEASUREMENTS

The interference experiment with deterministic absorption is obviously not a joint measurement in the strict sense of the word, since path and interference are at no time detected jointly. In fact, it can be seen as a realization of a proposal by Abu-Zeid.[12] Such "joint" experiments may be called *either/or* measurements, to distinguish them from real joint measurements. They are not very interesting as joint measurements. Weaker than the notion of an "either/or measurement," is the notion of *trivial* joint measurement. We shall call a joint nonideal measurement, described by the bivariate POVM $\{\underline{R}_{mn}\}$, a trivial joint measurement of the two observables $\{\underline{M}_k\}$ and $\{\underline{N}_l\}$ if it does not yield more information, i.e., if $\forall_{m,n}$ $\mathrm{Tr}[(\underline{\rho}-\underline{\rho}')\underline{R}_{mn}]=0$ for all density operators $\underline{\rho}$ and $\underline{\rho}'$ for which $\forall_{k,l}$ $\mathrm{Tr}[(\underline{\rho}-\underline{\rho}')\underline{M}_k] = \mathrm{Tr}[(\underline{\rho}-\underline{\rho}')\underline{N}_l] = 0$. It is straightforward to prove that the neutron-interference experiments, both with stochastic and with deterministic absorption, are trivial joint measurements of the interference and path observables. This can, for stochastic absorption, most easily be seen from (14) and (15), which demonstrate that $\mathrm{Tr}(\underline{\rho}\underline{M}_i)$ are determined completely by $\mathrm{Tr}(\underline{\rho}\underline{P}_+)$ and $\mathrm{Tr}(\underline{\rho}\underline{P}_A)$. For deterministic absorption this is a direct consequence of the either/or nature of the experiment.

One reason for trying to measure two observables jointly instead of separately would be the possibility of estimating correlation between the observables, because information about this correlation would not seem to be present in the separate measurements. It is easy to see that the values of $\mathrm{Tr}(\underline{\rho}\underline{P}_+)$ and $\mathrm{Tr}(\underline{\rho}\underline{P}_A)$ do not determine the density operator $\underline{\rho}$ completely. Hence we might presume that the correlation between interference and path observables could be left undetermined by the separate measurements. If this is the case, however, then the interference measurements with absorption do not seem to perform better in this respect. From the algebraic structure of the equations giving the relation between the POVM's $\{\underline{R}_{mn}\}$ on the one hand, and $\{\underline{M}_k\}$ and $\{N_l\}$ on the other, it seems necessary that, in order to have a nontrivial joint measurement, it must be described by a POVM $\{\underline{R}_{mn}\}$ consisting of at least four positive operators, which are independent apart from the condition (1). One possibility to achieve this is by inserting a second absorber into the interferometer, yielding also a probability for the neutron to be absorbed if it takes the other path. Unfortunately this scheme does not work. Although this experiment also is a joint nonideal measurement of path and interference, it is a trivial one due to the mutual dependence of the two absorption probabilities.

In order to implement a true four-operator POVM, we consider a setup (Fig. 4) inspired by Ref. 2 (see also Ref. 11). Here we assume the input beams to be polarized. After the first beamsplitter and the phase shifter, an rf spin-rotating device is inserted, which is intended to rotate the neutron's spin by 2π. The idea is that the spin rotation is accompanied by a backaction on the electromagnetic (em) field, e.g., by an exchange of one or more photons. If a measurement of photon number is made on the em field, one may hope to detect whether or not such a rotation has taken place. If we start with one input beam, the transformation is schematically given by

$$
\begin{aligned}
|L\rangle\otimes|\zeta\rangle &\rightarrow \tfrac{1}{2}\sqrt{2}(i|L\rangle+|R\rangle)\otimes|\zeta\rangle \rightarrow \tfrac{1}{2}\sqrt{2}[i|L\rangle+\exp(i\chi)|R\rangle]\otimes|\zeta\rangle \\
&\rightarrow \tfrac{1}{2}i\sqrt{2}[i|L\rangle\otimes|\zeta\rangle-\exp(i\chi)|R\rangle\otimes|\xi\rangle] \\
&\rightarrow \tfrac{1}{2}i[(-|L\rangle+i|R\rangle)\otimes|\zeta\rangle-\exp(i\chi)(|L\rangle+i|R\rangle)\otimes|\xi\rangle] \; . \qquad (23)
\end{aligned}
$$

Here $|\zeta\rangle$ and $|\xi\rangle$ indicate the states of the em field before and after a spin rotation, respectively. We measure jointly on the space of final neutron states the PVM corresponding to detectors D_A and D_B, and on the photon space a POVM $\{\underline{N}_+,\underline{N}_-\}$. This can be seen (for fixed $|\zeta\rangle$) as a joint measurement on the initial neutron state. The joint POVM with respect to the neutron input state $a|L\rangle+b|R\rangle$ reads

$$
\begin{aligned}
\underline{M}_{A\pm} &= \tfrac{1}{4}\langle\zeta|\underline{N}_\pm|\zeta\rangle \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \\
&+ \tfrac{1}{4}\langle\zeta|\underline{N}_\pm|\xi\rangle \exp(i\chi) \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \\
&+ \tfrac{1}{4}\langle\xi|\underline{N}_\pm|\zeta\rangle \exp(-i\chi) \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\
&+ \tfrac{1}{4}\langle\xi|\underline{N}_\pm|\xi\rangle \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} . \qquad (24)
\end{aligned}
$$

For the elements $\underline{M}_{B\pm}$ we get an analogous expression, with an extra minus sign in the second and third term. Then it can easily be shown that the marginals of the POVM (24) are

$$
\begin{aligned}
&\begin{pmatrix} \underline{M}_{A+}+\underline{M}_{B+} \\ \underline{M}_{A-}+\underline{M}_{B-} \end{pmatrix} = (\lambda_{mk}) \begin{pmatrix} \underline{P}_+ \\ \underline{P}_- \end{pmatrix} , \\
&(\lambda_{mk}) = \begin{pmatrix} \langle\zeta|\underline{N}_+|\zeta\rangle & \langle\xi|\underline{N}_+|\xi\rangle \\ \langle\zeta|\underline{N}_-|\zeta\rangle & \langle\xi|\underline{N}_-|\xi\rangle \end{pmatrix} , \\
&\begin{pmatrix} \underline{M}_{A+}+\underline{M}_{A-} \\ \underline{M}_{B+}+\underline{M}_{B-} \end{pmatrix} = (\mu_{nl}) \begin{pmatrix} \underline{\tilde{P}}_A \\ \underline{\tilde{P}}_B \end{pmatrix} , \\
&(\mu_{nl}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + |\langle\zeta|\xi\rangle| \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} .
\end{aligned} \qquad (25)
$$

Here $\langle\zeta|\xi\rangle = |\langle\zeta|\xi\rangle|\exp(i\theta)$, and $\{\underline{\tilde{P}}_A,\underline{\tilde{P}}_B\}$ is the PVM corresponding to the interference observable with phase $\chi+\theta$. From (25) it follows that the accuracy of the path measurement is determined by the extent to which the observable $\{\underline{N}_+,\underline{N}_-\}$ distinguishes the states $|\zeta\rangle$ and $|\xi\rangle$. If $|\zeta\rangle$ is a coherent state, then taking the photon number as the observable would not be very effective in this respect. A more effective observable can be found by applying a criterion discussed by Uffink and Hilgevoord.[11] Without loss of generality we may introduce two orthogonal vectors $|\eta\rangle$ and $|\eta^\perp\rangle$ such that

$$
\begin{aligned}
|\zeta\rangle &= \sqrt{1-\gamma}\,|\eta\rangle+\sqrt{\gamma}\exp(i\tau)|\eta^\perp\rangle , \\
|\xi\rangle &= \exp(i\theta)[\sqrt{\gamma}\exp(i\tau)|\eta\rangle+\sqrt{1-\gamma}\,|\eta^\perp\rangle]
\end{aligned} \qquad (26)
$$

(where $0\le\gamma\le\frac{1}{2}$). Then the PVM $\{\underline{N}_+,\underline{N}_-\}$, defined by

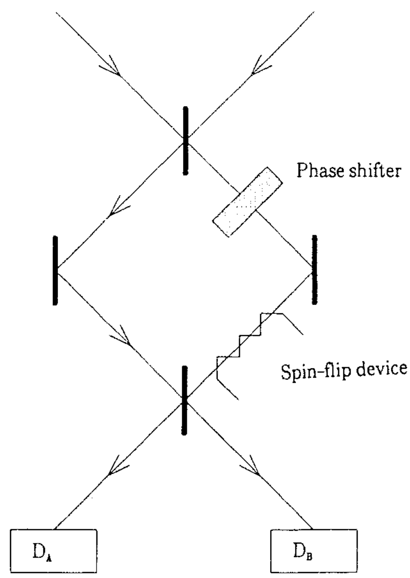

FIG. 4. Interferometer with spin flip. Neutrons in the right partial beam undergo a double spin flip. The influence of the spin flip on the em field may be used to effect a QND detection of the neutron's passing.

$\underline{N}_+ = 1 - \underline{N}_- = |\eta\rangle\langle\eta|$, optimally distinguishes $|\zeta\rangle$ and $|\xi\rangle$ in the sense of the mentioned criterion. Furthermore,

$$\langle\zeta|\xi\rangle = 2\sqrt{\gamma(1-\gamma)}\cos(\tau)\exp(i\theta) \qquad (27)$$

and

$$\langle\zeta|\underline{N}_+|\zeta\rangle = 1-\gamma, \ldots , \qquad (28)$$

enabling us to express the nonideality matrices of (25) in terms of the parameters γ and τ. For the amounts (4b) of nonideality we get

$$\epsilon_{(\lambda)} = 2\gamma, \quad \epsilon_{(\mu)} = 1 - 2\sqrt{\gamma(1-\gamma)}\cos(\tau) . \qquad (29)$$

In Fig. 3(b) (dotted line) this is plotted for $\tau=0$, which is the value of τ for which the $\epsilon_{(\lambda)} - \epsilon_{(\mu)}$ track is approaching the value $\epsilon_{(\lambda)} = \epsilon_{(\mu)} = 0$ most closely. In Fig. 3(a) $\delta_{(\lambda)}$ versus $\delta_{(\mu)}$ is plotted. It is seen also here that this QND scheme is in principle capable of approaching the quantum limit more closely than either of the earlier methods. Note that no interference pattern can be detected at all if $\langle\xi|\underline{N}_\pm|\zeta\rangle = 0$.

WIGNER MEASURE

The information regarding the joint nonideal measurement described by the bivariate POVM $\{\underline{R}_{mn}\}$ of (3) can, alternatively, be stored in the operator-valued Wigner measure defined by

$$\underline{W}_{kl} = \sum_{m,n} (\lambda^{-1})_{km} (\mu^{-1})_{ln} \underline{R}_{mn} , \qquad (30)$$

and satisfying

$$\sum_l \underline{W}_{kl} = \underline{M}_k, \quad \sum_k \underline{W}_{kl} = \underline{N}_l . \qquad (31)$$

For the POVM (13) and the stochastic matrices given in (14) and (15) we obtain

$$\underline{W}_{11} = \frac{1}{4}\begin{bmatrix} 1-2\cos(\chi) & -i+2\sin(\chi) \\ i+2\sin(\chi) & 1+2\cos(\chi) \end{bmatrix},$$
$$\underline{W}_{12} = \frac{1}{4}\begin{bmatrix} 1+2\cos(\chi) & -i-2\sin(\chi) \\ i-2\sin(\chi) & 1-2\cos(\chi) \end{bmatrix}, \qquad (32)$$
$$\underline{W}_{21} = \underline{W}_{22} = \frac{1}{4}\cdot\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} .$$

It can easily be verified that the expectation values $\mathrm{Tr}(\rho\underline{W}_{ij})$ do not determine the density operator completely. This is due to the linear dependency $\underline{W}_{11} + \underline{W}_{12} = 2\underline{W}_{21}$ satisfied by this Wigner measure. The Wigner measure resulting from (24) does separate the states [i.e, the POVM (24) is *informationally complete*[13,14]] if the photon observable $\{\underline{N}_+, \underline{N}_-\}$ is taken appropriately. Thus it can easily be seen that the choice $\underline{N}_+ = |\eta\rangle\langle\eta|$ as defined by (26) yields informational completeness if $\tau \neq 0$. The example demonstrates that the concept of a Wigner distribution may have a wider application than the usual phase-space representation of quantum mechanics. It is interesting to notice that the observable $\{\underline{N}_+, \underline{N}_-\}$ distinguishes maximally between the state $|\zeta\rangle$ and $\xi\rangle$ if $\tau=0$. This demonstrates the fact that a measurement that is optimal in one respect need not yield maximal information.

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